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Classical and quantum measurements of position

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Abstract. We study the dynamics of classical and quantum systems undergoing a continuous measurement of position by schematizing the measurement apparatus with an infinite set of harmonic oscillators at finite temperature linearly coupled to the measured system. Selective and non-selective measurement processes are then introduced according to a selection of or an average over all possible initial configurations of the measurement apparatus. At quantum level, the selective processes are described by a nonlinear stochastic Schrödinger equation whose solutions evolve into properly defined coherent states in the case of linear systems. For arbitrary measured systems, classical behaviour is always recovered in the macroscopic limit.

1. Introduction

A fundamental problem in quantum mechanics is the relationship between the states in the Hilbert space of a quantum system and the states in the phase space of the corresponding classical system. This is particularly evident in the case of a superposition of states which are individually mapped in the macroscopic limit, formally $\hbar \to 0$, into distinguishable classical states [1, 2].

One important step toward the solution of this problem has been made by recognizing that a system is never completely isolated by the external world. It has been argued that an external environment can, after a transient whose duration presumably depends on the coupling strength, drive the totality of the admissible states of the Hilbert space into those having macroscopic limit [3–5].

Among all the conceivable situations which require the interaction with an environment, a peculiar role is played by the measurement processes. Indeed, whenever any physical property of a system is investigated, an unavoidable coupling with the degrees of freedom of the measurement apparatus must be invoked. Taking into account these external degrees of freedom naturally provides a generalization of the von Neumann postulate [6] to continuous measurements [7–9].

There exists a basic difference between a general environment and one schematizing a measurement process. According to the Copenhagen interpretation, classical behaviour of the measurement apparatus has to be assumed before the information is registered by the

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observer [5, 10, 11]. This requirement, although establishing a classical connection between observer and meter, does not imply a classical behaviour of the observed system since its interaction with the meter fully preserves quantum features. This aspect may be emphasized by comparing the coupling of the same measurement apparatus, classically controlled by the observer, to classical or quantum systems.

This paper is devoted to establishing the relationship between the dynamics of classical and quantum systems under the influence of a measurement process. We model the effect of the measurement by allowing the measured system to linearly interact with an infinite set of harmonic oscillators. The interaction occurs in the configuration space thus representing a direct measurement of position. Our approach shouldn't be understood as the modelling of a particular device, e.g., a cloud chamber in which the oscillators represent molecules distributed in a medium [12]. Our oscillators are the modes of an interaction field which reproduces the main features expected in a measurement process, e.g., the wavefunction collapse, without *ad hoc* rules. In contrast to other abstract models [13], however, our approach has a certain degree of realism, since it allows us to define the reading of the measurement results operatively.

The oscillators representing the measurement device are chosen at thermal equilibrium with temperature T and continuously distributed in frequency with a proper density. These conditions ensure that, for a chosen initial configuration of the oscillators, the classical measured system is described by a Markovian Langevin equation with white noise and a certain relaxation time γ^{-1}. The constant γ, which fixes the magnitude of the density of oscillators, represents the strength of the measurement process. In the quantum case, a further characteristic time of the measurement apparatus, $\hbar/k_B T$, arises. The requirement of classical behaviour of the meter with respect to the observer implies that a high-temperature condition must hold, so that this thermal fluctuation time must be much shorter than the relaxation one. As a consequence, the measured quantum system is described by a stochastic nonlinear Schrödinger equation [14–17].

Details of the derivation of the stochastic equations describing classical and quantum systems during a measurement process are given in sections 2 and 3, respectively. We call these processes selective, since they correspond to a single measurement act with initial conditions of the meter selected from those compatible with the assumed thermal equilibrium. Alternatively, one can also consider non-selective measurement processes corresponding to an average over all initial configurations of the meter. In this case, the dynamics of the classical and quantum measured systems is described by a Fokker–Plank equation and a trace-preserving positive master equation, respectively. The experimental results obtained by repeating a measurement on a system always in the same initial state or performing the same measurement on an ensemble of equally prepared independent systems is directly comparable with the solution of these non-selective equations. Equivalent results are predicted by averaging the solutions of the selective equations, classical or quantum, over the realizations of the corresponding stochastic process.

In section 4 we show how to infer the measurement results by the reading of an appropriate pointer. Since the oscillators representing the meter are classical with respect to the observer, the pointer can be defined in terms of the coordinates of these oscillators which, in turn, reflect the status of the measured system.

As for closed quantum systems, in the presence of a measurement process there also exists a class of states, namely the coherent ones, which admits, in a proper sense, the $\hbar \to 0$ limit. These coherent states, explicitly built in section 5, are Gaussian localized states in the co-moving frame of a measured linear system [15, 18, 19]. In section 6 we show that for linear systems the solutions of the stochastic Schrödinger equation converge to a coherent state localized around a point in the phase space which moves according to a Langevin-like equation. This equation reduces to the classical one for $\hbar \to 0$. The convergence into this coherent state occurs in a timescale $(\hbar/\gamma k_B T)^{1/2}$, the geometric mean of the two characteristic times associated with the measurement apparatus. This time diverges for an unmeasured system ($\gamma \to 0$) and vanishes in the macroscopic limit.

In the case of nonlinear systems, the phase-space localization through convergence into a coherent state becomes the leading dynamical process as $\hbar \to 0$. This is sufficient to demonstrate that classical behaviour is always recovered in the macroscopic limit, avoiding any paradoxical quantum feature.

2. Classical systems

Let us consider a system described by the Hamiltonian

$$H(p, q, t) = \frac{p^2}{2m} + V(q, t) \tag{2.1}$$

and suppose that we want to measure its position q ($q \in \mathbb{R}$, for simplicity). We schematize the measurement process by the interaction of the system (2.1) with a set of particles of mass M and canonical coordinates (P_n, Q_n) via a harmonic potential which, for evident physical reasons, must depend on the relative distances $Q_n - q$. The Hamiltonian for the total system is then taken as

$$H_{tot} = H(p, q, t) + H_m(P, Q - q) \tag{2.2}$$

where

$$H_m(P, Q - q) = \sum_n \left[\frac{P_n^2}{2M} + \frac{M\omega_n^2}{2} (Q_n - q)^2 \right] \tag{2.3}$$

represents the measurement apparatus linearly coupled to the measured system. Here and in the following, Q and P are a shortening for the whole set $\{Q_n\}$ and $\{P_n\}$. Note that the Hamiltonian (2.3) can be interpreted as that of a set of harmonic oscillators with equilibrium positions $Q_n = q$. Our model is thus reduced to the exactly solvable problem of a system interacting with a bath of harmonic oscillators [20] but with the difference that the interaction potential is invariant under space translation. The importance of this invariance in avoiding the appearance of infinite renormalization potentials has been recently underlined in [21] in the framework of the classical/quantum Brownian motion. We stress that the long-range nature of the quadratic potential in (2.3) prevents us from considering a situation in which the measured system does not interact with the meter, so that our model cannot take into account the switching-on of the measurement.

At the classical level, the Newton equation for each harmonic oscillator can be solved explicitly in terms of the function $q(t)$ and the values of the coordinates $Q_n' = Q_n(t')$, $P_n' = P_n(t')$ and $q' = q(t')$ at the initial time t'

$$Q_n(t) = (Q_n' - q') \cos[\omega_n(t - t')] + \frac{P_n'}{M\omega_n} \sin[\omega_n(t - t')]$$
$$+ q(t) - \int_{t'}^{t} \mathrm{d}s \ \cos[\omega_n(t - s)] \, \dot{q}(s). \tag{2.4}$$

We will also assume $p(t') = p'$. When this solution is inserted in the Newton equation for the measured system, the following equation for $q(t)$ is obtained:

$$m\ddot{q}(t) + \int_{t'}^{t} \mathrm{d}s \ \Gamma(t - s)\dot{q}(s) + \partial_q V(q(t), t) = \Pi(t) \tag{2.5}$$

where

$$\Gamma(t-s) = \sum_n M\omega_n^2 \cos[\omega_n(t-s)] \tag{2.6}$$

and

$$\Pi(t) = \sum_n M\omega_n^2 \left\{ (Q'_n - q')\cos[\omega_n(t-t')] + \frac{P'_n}{M\omega_n}\sin[\omega_n(t-t')] \right\}. \tag{2.7}$$

In the limit of an infinite number of oscillators, if the corresponding initial conditions Q' and P' are a realization of a stochastic process in the phase space, then $\Pi(t)$ is a realization of a stochastic process in time. If, as we will suppose, the oscillators are at thermal equilibrium with temperature T, the initial conditions to be considered are typical realizations of the stochastic process corresponding to the equilibrium Gibbs measure [22]. In this case, the following statistical properties for $\Pi(t)$ hold:

$$\overline{\Pi(t)} = 0 \qquad \overline{\Pi(t)\Pi(s)} = k_B T\, \Gamma(t-s) \tag{2.8}$$

where

$$\overline{\cdots} = \frac{\int \mathrm{d}P'\,\mathrm{d}Q' \cdots \exp\big(-H_m(P', Q'-q')/k_B T\big)}{\int \mathrm{d}P'\,\mathrm{d}Q' \exp\big(-H_m(P', Q'-q')/k_B T\big)}. \tag{2.9}$$

Note that the initial conditions corresponding to the definition (2.9) respect the space translation symmetry of the Hamiltonian H_m thus implying, in agreement with the long-range nature of the quadratic potential, a correlation between the coordinates Q' and q'.

The friction term in the stochastic differential equation (2.5) contains memory effects which are an unessential complication in our context. A Markovian evolution can be obtained by choosing an appropriate continuous distribution of the frequencies $\{\omega_n\}$. For a frequency density

$$\frac{\mathrm{d}N}{\mathrm{d}\omega} = \frac{2m\gamma}{\pi M\omega^2}\theta(\Omega-\omega) \tag{2.10}$$

where $\theta(x)$ is 1 for $x > 0$ and 0 otherwise, we obtain

$$\Gamma(t-s) = \int_0^\infty \mathrm{d}\omega\, \frac{\mathrm{d}N}{\mathrm{d}\omega} M\omega^2 \cos[\omega(t-s)] = 2m\gamma \frac{\sin[\Omega(t-s)]}{\pi(t-s)} \simeq 2m\gamma\,\delta(t-s). \tag{2.11}$$

The approximation holds for $t - s \gg \Omega^{-1}$. For $\Omega^{-1} \ll \tau$, where τ is the fastest time scale at which the measured system evolves, $\Pi(t)$ can be approximated with a white noise and equation (2.5) rewritten as

$$\mathrm{d}p(t) = -\big[\gamma p(t) + \partial_q V(q(t), t)\big]\,\mathrm{d}t + \sqrt{2m\gamma k_B T}\,\mathrm{d}w(t) \tag{2.12}$$

$$\mathrm{d}q(t) = \frac{p(t)}{m}\,\mathrm{d}t. \tag{2.13}$$

Here, we introduced the Wiener process $\mathrm{d}w(t) = (2m\gamma k_B T)^{-1/2}\Pi(t)\,\mathrm{d}t$ having zero average, $\overline{\mathrm{d}w(t)} = 0$, and standard scaling, $\overline{\mathrm{d}w(t)\,\mathrm{d}w(t)} = \mathrm{d}t$. Note that the friction coefficient γ and the temperature T completely define the fluctuation and dissipation phenomena induced by the interaction with the measurement apparatus.

Equations (2.12) and (2.13) describe the evolution of the system during a selective measurement, i.e. a measurement in which a realization of the stochastic process $p(t)$, $q(t)$ is selected according to certain initial conditions of the measurement apparatus. Alternatively, one can consider a non-selective measurement corresponding to an average over all possible realizations of the stochastic process. In this case, the measured system is described by a probability density $W(p, q, t)$ which is determined by the Fokker–Plank equation [23] associated with (2.12) and (2.13):

$$\partial_t W(p, q, t) = \left[-\frac{p}{m} \partial_q + \partial_q V(q, t) \partial_p + \partial_p \left(\gamma p + m\gamma k_B T \partial_p \right) \right] W(p, q, t) \tag{2.14}$$

with initial conditions $W(p, q, t') = \delta(p - p')\delta(q - q')$. The probability density $W(p, q, t)$ allows us to directly evaluate averages of any function of $p(t)$ and $q(t)$. In particular, for the position we have the average value

$$\overline{q(t)} = \int \mathrm{d}p\ \mathrm{d}q\ W(p, q, t)\, q \tag{2.15}$$

with a variance

$$\Delta q^2(t) = \int \mathrm{d}p\ \mathrm{d}q\ W(p, q, t) \left[q - \overline{q(t)}\, \right]^2 = \overline{q(t)^2} - \overline{q(t)}^2. \tag{2.16}$$

For $\gamma \to 0$ the effect of the measurement vanishes. In this case, equation (2.14) becomes the Liouville equation for the isolated system (2.1), the average value (2.15) gives the corresponding time-dependent solution, and the variance (2.16) vanishes.

3. Quantum systems

At the quantum level, the measured system is conveniently described by a reduced density matrix obtained by tracing out the coordinates of the measurement apparatus in the total density matrix

$$\varrho(q_1, q_2, t) = \int \mathrm{d}Q\ \varrho_{\mathrm{tot}}(q_1, Q, q_2, Q, t). \tag{3.1}$$

We assume that at the initial time t' the oscillators of the measurement apparatus are at thermal equilibrium with temperature T and the total density matrix is factorized as

$$\varrho_{\mathrm{tot}}(q_1', Q_1', q_2', Q_2', t') = \varrho(q_1', q_2', t')\varrho_m(\delta Q_1', \delta Q_2', t') \tag{3.2}$$

where

$$\varrho_m(\delta Q_1', \delta Q_2', t') = \prod_n \sqrt{\frac{M\omega_n}{\pi\hbar} \tanh\left(\frac{\hbar\omega_n}{2k_B T}\right)} \times \exp\left[-\frac{M\omega_n}{2\hbar} \left(\frac{\delta Q_{1n}'^2 + \delta Q_{2n}'^2}{\tanh(\hbar\omega_n / k_B T)} - \frac{2\delta Q_{1n}' \delta Q_{2n}'}{\sinh(\hbar\omega_n / k_B T)} \right) \right] \tag{3.3}$$

with $\delta Q_1' = Q_1' - Q_{\mathrm{eq}}'$ and $\delta Q_2' = Q_2' - Q_{\mathrm{eq}}'$. The requirement of translational and time reversal invariance [21] implies

$$Q_{\mathrm{eq}}' = \frac{q_1' + q_2'}{2}. \tag{3.4}$$

Analogously to the classical case, this choice corresponds to an initial condition (3.2) factorized but correlated.

At a later time t, the reduced density matrix is obtained through a Green function

$$\varrho(q_1, q_2, t) = \int \mathrm{d}q_1'\ \mathrm{d}q_2'\ G(q_1, q_2, t; q_1', q_2', t')\varrho(q_1', q_2', t') \tag{3.5}$$

whose path-integral representation

$$G(q_1, q_2, t; q_1', q_2', t') = \int d[p_1]\, d[q_1]_{q_1',t'}^{q_1,t} \int d[p_2]\, d[q_2]_{q_2',t'}^{q_2,t} \times \exp\left\{ \frac{i}{\hbar} S[p_1, q_1] - \frac{i}{\hbar} S[p_2, q_2] - Z[p_1, q_1, p_2, q_2] \right\} \qquad (3.6)$$

is the free evolution of the measured system modified by the influence functional

$$\exp\{-Z[p_1, q_1, p_2, q_2]\} = \int dQ \int dQ_1'\, dQ_2' \int d[P_1]\, d[Q_1]_{Q_1',t'}^{Q,t} \int d[P_2]\, d[Q_2]_{Q_2',t'}^{Q,t} \times \exp\left\{ \frac{i}{\hbar} S_m[P_1, Q_1 - q_1] - \frac{i}{\hbar} S_m[P_2, Q_2 - q_2] \right\} \varrho_m(\delta Q_1', \delta Q_2', t'). \qquad (3.7)$$

Here, $S[p, q]$ and $S_m[P, Q - q]$ are the classical actions corresponding to the Hamiltonians H and H_m, respectively:

$$S[p, q] = \int_{t'}^{t} ds\, \left[p(s)\dot{q}(s) - H(p, q, s) \right] \qquad (3.8)$$

$$S_m[P, Q - q] = \int_{t'}^{t} ds\, \left[P(s)\dot{Q}(s) - H_m(P, Q - q, s) \right]. \qquad (3.9)$$

The functional measure with boundary conditions $q(t') = q'$ and $q(t) = q$ is obtained by slicing the interval $[t', t]$ at times $t^{(n)} = t' + (t - t')n/N$, $n = 1, \ldots, N$, and taking the $N \to \infty$ limit. All time integrals can be approximated by sums:

$$\int_{t'}^{t} ds\, f(p, q, s) = \sum_{n=1}^{N} \int_{t^{(n-1)}}^{t^{(n)}} ds\, f(p, q, s) \simeq \frac{t - t'}{N} \sum_{n=1}^{N} f\left(p^{(n)}, q^{(n)}, t^{(n)}\right) \qquad (3.10)$$

where $p^{(n)} = p(t^{(n)})$, $q^{(n)} = q(t^{(n)})$ and

$$d[p]\, d[q]_{q',t'}^{q,t} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{dp^{(n)}}{2\pi\hbar} \prod_{n=1}^{N-1} dq^{(n)}. \qquad (3.11)$$

Analogous relations hold for $d[P]\, d[Q]_{Q',t'}^{Q,t}$.

The influence functional (3.7) contains only Gaussian integrals and can be evaluated exactly. The result is [24]

$$Z[p_1, q_1, p_2, q_2] = \int_{t'}^{t} ds \int_{t'}^{s} du\, \left[q_1(s) - q_2(s) \right] \left\{ \Lambda(s - u)\left[q_1(u) - q_2(u) \right] + \frac{i}{2m\hbar} \Gamma(s - u)\left[p_1(u) + p_2(u) \right] \right\} \qquad (3.12)$$

where

$$\Lambda(s - u) = \frac{M}{2\hbar} \sum_n \omega_n^3 \coth\left(\frac{\hbar\omega_n}{2k_B T} \right) \cos[\omega_n(s - u)] \qquad (3.13)$$

and $\Gamma(s - u)$, defined by (2.6), are called fluctuation and dissipation kernels, respectively [20]. The double time integral which appears in equation (3.12) is responsible for memory effects which break the semi-group property of the evolution of the measured system, i.e.

$$G(q_1, q_2, t; q_1', q_2', t') = \int dq_1''\, dq_2''\, G(q_1, q_2, t; q_1'', q_2'', t'')\, G(q_1'', q_2'', t''; q_1', q_2', t') \qquad (3.14)$$

for $t' < t'' < t$. Two are the sources of this non-Markovian behaviour. In the dissipation kernel Γ, the origin of memory effects is classical and Markovian behaviour is obtained if one assumes the frequency distribution (2.10) with $\Omega^{-1} \ll \tau$. In the fluctuation kernel Λ, the assumption of these conditions does not remove the memory effects due to the quantum behaviour of the oscillators in the thermal bath. However, in the framework of a theory of measurement processes, the measurement apparatus, which is an interface between the observer (classical) and the measured system (quantum), must have classical behaviour with respect to the former in order to avoid paradoxical features [5]. In the present model, the nature of the interaction between the measurement apparatus and the external environment is fixed by the ratios $\hbar\omega_n/k_B T$ so that the requirement of classical behaviour of the measurement apparatus with respect to the observer imposes the condition $k_B T \gg \hbar\Omega$.

For $\hbar/k_B T \ll \Omega^{-1} \ll \tau$ and frequency distribution (2.10), equation (3.12) can be approximated (see appendix A) by

$$Z[p_1, q_1, p_2, q_2] = \int_{t'}^{t} \mathrm{d}s \left\{ \frac{m\gamma k_B T}{\hbar^2} \big[q_1(s) - q_2(s)\big]^2 + \frac{\mathrm{i}\gamma}{2\hbar} \big[q_1(s) - q_2(s)\big]\big[p_1(s) + p_2(s)\big] \right\}. \tag{3.15}$$

A differential equation for the reduced density matrix operator can be then derived with standard methods. According to equation (3.15) and using the path-integral representation of G, we have

$$\begin{aligned} \varrho(q_1, q_2, t + \mathrm{d}t) &= \int \mathrm{d}q_1'\, \mathrm{d}q_2'\; G(q_1, q_2, t + \mathrm{d}t; q_1', q_2', t)\; \varrho(q_1', q_2', t) \\ &= \int \frac{\mathrm{d}p_1}{2\pi\hbar}\, \mathrm{d}q_1' \int \frac{\mathrm{d}p_2}{2\pi\hbar}\, \mathrm{d}q_2'\; \exp\bigg\{ \frac{\mathrm{i}}{\hbar} p_1\big(q_1 - q_1'\big) - \frac{\mathrm{i}}{\hbar} p_2\big(q_2 - q_2'\big) \\ &\quad + \bigg[-\frac{\mathrm{i}}{\hbar} H\big(p_1, q_1, t\big) + \frac{\mathrm{i}}{\hbar} H\big(p_2, q_2, t\big) - \frac{m\gamma k_B T}{\hbar^2}\big(q_1 - q_2\big)^2 \\ &\quad - \frac{\mathrm{i}\gamma}{2\hbar}\big(q_1 - q_2\big)\big(p_1 + p_2\big) \bigg] \mathrm{d}t \bigg\} \varrho\big(q_1', q_2', t\big). \end{aligned} \tag{3.16}$$

By using the identity $\langle q|p\rangle = (2\pi\hbar)^{-1/2} \exp(\mathrm{i}pq/\hbar)$ and expanding the exponential containing the infinitesimal time $\mathrm{d}t$, the above equation can be cast in the form

$$\langle q_1|\hat{\varrho}(t + \mathrm{d}t)|q_2\rangle = \langle q_1|\hat{\varrho}(t)|q_2\rangle + \langle q_1|\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}(t)|q_2\rangle \tag{3.17}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}(t) = -\frac{\mathrm{i}}{\hbar}\big[\hat{H}(\hat{p}, \hat{q}, t), \hat{\varrho}(t)\big] - \frac{m\gamma k_B T}{\hbar^2}\big[\hat{q}, \big[\hat{q}, \hat{\varrho}(t)\big]\big] - \frac{\mathrm{i}\gamma}{2\hbar}\big[\hat{q}, \big\{\hat{p}, \hat{\varrho}(t)\big\}\big]. \tag{3.18}$$

Note that $\mathrm{d}[\mathrm{Tr}\,\hat{\varrho}(t)]/\mathrm{d}t = 0$ so that we can assume $\mathrm{Tr}\,\hat{\varrho}(t) = 1$.

Equation (3.18) describes the evolution of the measured system with initial conditions of the oscillator system statistically distributed according to equation (3.3). This is a non-selective measurement process to be compared with the classical one described by equation (2.14).

The quantum–classical correspondence of non-selective measurements can be extended to selective processes. Introducing

$$A(p(t), q(t)) = \sqrt{\frac{2m\gamma k_B T}{\hbar^2}}\; q(t) + \mathrm{i}\sqrt{\frac{\gamma}{8mk_B T}}\; p(t) \tag{3.19}$$

$$B(p(t)) = \sqrt{\frac{\gamma}{8mk_{\rm B}T}}\; p(t) \tag{3.20}$$

equation (3.15) can be rewritten as

$$\begin{aligned} Z = \int_{t'}^{t} {\rm d}s \left[\frac{{\rm i}\gamma}{2\hbar}(q_1p_1 - q_2p_2) + \frac{1}{2}A_1A_1^* + \frac{1}{2}A_2A_2^* - A_1A_2^* - \frac{1}{2}(B_1 - B_2)^2 \right] \\ = \int_{t'}^{t} {\rm d}s \left[\frac{{\rm i}\gamma}{2\hbar}(q_1p_1 - q_2p_2) + \frac{1}{2}A_1A_1^* + \frac{1}{2}A_2A_2^* - \frac{1}{2}(A_1 - a)^2 \right. \\ - \frac{1}{2}(A_2^* - a^*)^2 + aa^* - A_1a^* - A_2^*a + \frac{1}{2}(A_1 - a - A_2^* + a^*)^2 \\ \left. - \frac{1}{2}(B_1 - b - B_2 + b)^2 \right] \end{aligned} \tag{3.21}$$

where A_i and B_i stand for $A(p_i, q_i)$ and $B(p_i)$, $i = 1, 2$, and $a(t)$ and $b(t)$ are two arbitrary functions, which are complex and real, respectively. The coupling between the components 1 and 2 of the system coordinates given by the last two squares of the exponential can be eliminated in terms of two functional integrations over white real noises by means of the identities

$$\begin{aligned} \exp\left[-\frac{1}{2}\int_{t'}^{t} {\rm d}s\; (A_1 - a - A_2^* + a^*)^2 \right] = \int {\rm d}[\xi]\; \exp\left\{ -\int_{t'}^{t} {\rm d}s\; \left[(A_1 - a)^2 + (A_2^* - a^*)^2 \right.\right. \\ \left.\left. - (A_1 - a)\xi - (A_2^* - a^*)\xi \right] \right\} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \exp\left[\frac{1}{2}\int_{t'}^{t} {\rm d}s\; (B_1 - b - B_2 + b)^2 \right] = \int {\rm d}[\eta]\; \exp\left\{ \int_{t'}^{t} {\rm d}s\; \left[(B_1 - b)^2 + (B_2 - b)^2 \right.\right. \\ \left.\left. + {\rm i}(B_1 - b)\eta + {\rm i}(B_2 - b)\eta \right] \right\}. \end{aligned} \tag{3.23}$$

Note that the above functional integration measures are Gaussian:

$${\rm d}[\xi] = \lim_{N\to\infty} \prod_{n=1}^{N} {\rm d}\xi^{(n)} \sqrt{\frac{t - t'}{2\pi N}} \exp\left(-\frac{(t - t')\xi^{(n)2}}{2N} \right) \tag{3.24}$$

so that

$$\overline{\xi(t)} = \int {\rm d}[\xi]\; \xi(t) = 0 \qquad \overline{\xi(t)\xi(s)} = \int {\rm d}[\xi]\; \xi(t)\,\xi(s) = \delta(t - s) \tag{3.25}$$

and analogously for η. The two-particle Green function (3.6) can be then rewritten in terms of a couple of one-particle Green functions

$$G(q_1, q_2, t; q_1', q_2', t') = \int {\rm d}[\xi]\;\; {\rm d}[\eta]\; G^+_{[\xi\eta]}(q_1, t; q_1', t')\, G^-_{[\xi\eta]}(q_2, t; q_2', t')^* \tag{3.26}$$

where

$$\begin{aligned} G^{\pm}_{[\xi\eta]}(q, t; q', t') = \int {\rm d}[p]\; {\rm d}[q]^{q,t}_{q',t'}\;\; \exp\left\{ \frac{\rm i}{\hbar}S[p, q] - \frac{{\rm i}\gamma}{2\hbar}\int_{t'}^{t} {\rm d}s\;\; pq \right. \\ + \int_{t'}^{t} {\rm d}s \left[-\frac{1}{2}AA^* - \frac{1}{2}(A - a)^2 - \frac{1}{2}aa^* + Aa^* + (A - a)\xi \right. \\ \left.\left. + (B - b)^2 \pm {\rm i}(B - b)\eta \right] \right\}. \end{aligned} \tag{3.27}$$

Assuming that the system is initially in a pure state, i.e. $\hat{\varrho}(t') = |\psi(t')\rangle\langle\psi(t')|$, at a later time t the reduced density matrix operator is expressed as a functional integral over pure states

$$\hat{\varrho}(t) = \int \mathrm{d}[\xi]\,\mathrm{d}[\eta]\,|\psi^{+}_{[\xi\eta]}(t)\rangle\langle\psi^{-}_{[\xi\eta]}(t)| \tag{3.28}$$

obtained by propagating $|\psi(t')\rangle$ with $G^{\pm}_{[\xi\eta]}$:

$$\langle q|\psi^{\pm}_{[\xi\eta]}(t)\rangle = \int \mathrm{d}q'\; G^{\pm}_{[\xi\eta]}(q,t;q',t')\;\langle q'|\psi(t')\rangle. \tag{3.29}$$

An evolution equation for the states $|\psi^{\pm}_{[\xi\eta]}(t)\rangle$ can be obtained by writing the explicit form of the propagators $G^{\pm}_{[\xi\eta]}$ between the times t and $t+\mathrm{d}t$

$$\begin{aligned}\langle q|\psi^{\pm}_{[\xi\eta]}(t+\mathrm{d}t)\rangle &= \int \mathrm{d}q'\; G^{\pm}_{[\xi\eta]}(q,t+\mathrm{d}t;q',t)\;\langle q'|\psi(t)\rangle\\
&= \int \frac{\mathrm{d}p}{2\pi\hbar}\mathrm{d}q'\;\exp\left\{\frac{\mathrm{i}}{\hbar}p\big(q-q'\big) + \left[-\frac{\mathrm{i}}{\hbar}H\big(p,q,t\big) - \frac{\mathrm{i}\gamma}{2\hbar}pq\right.\right.\\
&\quad - \frac{1}{2}A(p,q)A(p,q)^* - \frac{1}{2}[A(p,q)-a(t)]^2 - \frac{1}{2}a(t)a(t)^* + A(p,q)a(t)^*\\
&\quad \left.\left.+ [A(p,q)-a(t)]\xi(t) + [B(p)-b(t)]^2 \pm \mathrm{i}[B(p)-b(t)]\eta(t)\right]\mathrm{d}t\right\}\\
&\quad \times \langle q'|\psi^{\pm}_{[\xi\eta]}(t)\rangle.\end{aligned} \tag{3.30}$$

By using the identity $\langle q|p\rangle = (2\pi\hbar)^{-1/2}\exp(\mathrm{i}pq/\hbar)$ and expanding the exponential containing the Wiener processes $\mathrm{d}w_\xi(t) = \xi(t)\,\mathrm{d}t$ and $\mathrm{d}w_\eta(t) = \eta(t)\,\mathrm{d}t$ according to the Ito rule [25], we obtain the following stochastic differential equation†:

$$\begin{aligned}\mathrm{d}|\psi^{\pm}_{[\xi\eta]}(t)\rangle &= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}(\hat{p},\hat{q},t) + \frac{\gamma}{4}(\hat{p}\hat{q}+\hat{q}\hat{p})\right]|\psi^{\pm}_{[\xi\eta]}(t)\rangle\,\mathrm{d}t\\
&\quad - \frac{1}{2}\big[\hat{A}^\dagger\hat{A} + a(t)^*a(t) - 2a(t)^*\hat{A}\big]|\psi^{\pm}_{[\xi\eta]}(t)\rangle\,\mathrm{d}t\\
&\quad + \big[\hat{A} - a(t)\big]|\psi^{\pm}_{[\xi\eta]}(t)\rangle\,\mathrm{d}w_\xi(t) + \frac{1}{2}\big[\hat{B}-b(t)\big]^2|\psi^{\pm}_{[\xi\eta]}(t)\rangle\,\mathrm{d}t\\
&\quad \pm\mathrm{i}\big[\hat{B}-b(t)\big]|\psi^{\pm}_{[\xi\eta]}(t)\rangle\,\mathrm{d}w_\eta(t).\end{aligned} \tag{3.31}$$

The normalization condition for the reduced density matrix operator

$$\mathrm{Tr}\,\hat{\varrho}(t) = \int \mathrm{d}[\xi]\,\mathrm{d}[\eta]\,\langle\psi^{-}_{[\xi\eta]}(t)|\psi^{+}_{[\xi\eta]}(t)\rangle = 1 \qquad \int \mathrm{d}[\xi]\,\mathrm{d}[\eta] = 1 \tag{3.32}$$

is satisfied by imposing $\langle\psi^{-}_{[\xi\eta]}(t)|\psi^{+}_{[\xi\eta]}(t)\rangle = 1$. This fixes the arbitrary functions $a(t)$ and $b(t)$. Indeed, the requirement that the Ito differential

$$\begin{aligned}\mathrm{d}\langle\psi^{-}_{[\xi\eta]}(t)|\psi^{+}_{[\xi\eta]}(t)\rangle &= \langle\psi^{-}_{[\xi\eta]}(t)|\hat{A} - a(t) + \hat{A}^\dagger - a^*(t)|\psi^{+}_{[\xi\eta]}(t)\rangle\,\mathrm{d}w_\xi(t)\\
&\quad + 2\mathrm{i}\langle\psi^{-}_{[\xi\eta]}(t)|\hat{B} - b(t)|\psi^{+}_{[\xi\eta]}(t)\rangle\,\mathrm{d}w_\eta(t)\end{aligned} \tag{3.33}$$

† Note that $\langle q|p\rangle\big(-(\mathrm{i}\gamma/2\hbar)qp - \frac{1}{2}A(p,q)A(p,q)^*\big) = \langle q| - (\mathrm{i}\gamma/4\hbar)(\hat{q}\hat{p}+\hat{p}\hat{q}) - \frac{1}{2}\hat{A}^\dagger\hat{A}|p\rangle$.

vanishes implies

$$a(t) = \langle\psi^-_{[\xi\eta]}(t)|\hat{A}|\psi^+_{[\xi\eta]}(t)\rangle$$
$$= \sqrt{\frac{2m\gamma k_B T}{\hbar^2}}\ \langle\psi^-_{[\xi\eta]}(t)|\hat{q}|\psi^+_{[\xi\eta]}(t)\rangle + \mathrm{i}\sqrt{\frac{\gamma}{8mk_B T}}\ \langle\psi^-_{[\xi\eta]}(t)|\hat{p}|\psi^+_{[\xi\eta]}(t)\rangle \quad (3.34)$$

$$b(t) = \langle\psi^-_{[\xi\eta]}(t)|\hat{B}|\psi^+_{[\xi\eta]}(t)\rangle = \sqrt{\frac{\gamma}{8mk_B T}}\ \langle\psi^-_{[\xi\eta]}(t)|\hat{p}|\psi^+_{[\xi\eta]}(t)\rangle. \quad (3.35)$$

The appearance of two different Green functions $G^{\pm}_{[\xi\eta]}$ in (3.28) possibly introduces a violation of positivity in $\hat{\varrho}(t)$. This unphysical property is reflected in the anomalous definition of expectation values of the Hermitian operators, e.g., $\langle\psi^-_{[\xi\eta]}(t)|\hat{q}|\psi^+_{[\xi\eta]}(t)\rangle$, which may be complex. The problem is mathematically related to the presence of the B terms in (3.21). However, due to the high-temperature condition $\hbar/k_B T \ll \tau$ we have $B \ll A$ (see appendix A for details) and the last square in (3.21) can be neglected with respect to the last but one. In this case, equation (3.18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}(t) = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}(\hat{p},\hat{q},t),\hat{\varrho}(t)\right] - \frac{m\gamma k_B T}{\hbar^2}\left[\hat{q},\left[\hat{q},\hat{\varrho}(t)\right]\right] - \frac{\mathrm{i}\gamma}{2\hbar}\left[\hat{q},\left\{\hat{p},\hat{\varrho}(t)\right\}\right]$$
$$-\frac{\gamma}{16mk_B T}\left[\hat{p},\left[\hat{p},\hat{\varrho}(t)\right]\right]$$
$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{H}(\hat{p},\hat{q},t) + \frac{\gamma}{4}(\hat{p}\hat{q}+\hat{q}\hat{p}),\hat{\varrho}(t)\right] + \frac{1}{2}\left[\hat{A}\hat{\varrho}(t),\hat{A}^\dagger\right]$$
$$+\frac{1}{2}\left[\hat{A},\hat{\varrho}(t)\hat{A}^\dagger\right]. \quad (3.36)$$

This equation is of Lindblad class and provides a (completely) positive evolution of $\hat{\varrho}(t)$ [26]. The reduced density matrix operator can be decomposed in terms of a single state $|\psi_{[\xi]}(t)\rangle$ associated with the Green function $G_{[\xi]}$ obtained by neglecting the B terms in $|\psi^{\pm}_{[\xi\eta]}(t)\rangle$ and $G^{\pm}_{[\xi\eta]}$, respectively. Note the disappearance of the η noise. Equation (3.31) becomes the norm-preserving stochastic Schrödinger equation

$$\mathrm{d}|\psi_{[\xi]}(t)\rangle = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}(\hat{p},\hat{q},t) + \frac{\gamma}{4}(\hat{p}\hat{q}+\hat{q}\hat{p})\right]|\psi_{[\xi]}(t)\rangle\,\mathrm{d}t$$
$$-\frac{1}{2}\left[\hat{A}^\dagger\hat{A} + a(t)^*a(t) - 2a(t)^*\hat{A}\right]|\psi_{[\xi]}(t)\rangle\,\mathrm{d}t$$
$$+\left[\hat{A} - a(t)\right]|\psi_{[\xi]}(t)\rangle\,\mathrm{d}w_\xi(t) \quad (3.37)$$

with $a(t) = \langle\psi_{[\xi]}(t)|\hat{A}|\psi_{[\xi]}(t)\rangle$.

Equation (3.37) describes the evolution of the measured system for a realization of the stochastic processes $|\psi_{[\xi]}(t)\rangle$. This is a selective measurement process, which is related the non-selective one by the relationship

$$\hat{\rho}(t) = \overline{|\psi_{[\xi]}(t)\rangle\langle\psi_{[\xi]}(t)|} = \int \mathrm{d}[\xi]\ |\psi_{[\xi]}(t)\rangle\langle\psi_{[\xi]}(t)|. \quad (3.38)$$

The quantum expectation values of the observables, e.g. $q(t) = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$, are stochastic processes with average value

$$\overline{q(t)} = \int \mathrm{d}[\xi]\ \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle \quad (3.39)$$

and variance

$$\Delta q^2(t) = \int \mathrm{d}[\xi]\ \langle\psi_{[\xi]}(t)|\big[\hat{q} - \overline{q(t)}\,\big]^2|\psi_{[\xi]}(t)\rangle. \tag{3.40}$$

According to equation (3.38), these average quantities can be also directly computed by considering the non-selective measurement process described by equation (3.36)

$$\overline{q(t)} = \mathrm{Tr}\big[\hat{\varrho}(t)\hat{q}\big] = \int \mathrm{d}p\ \mathrm{d}q\ W(p,q,t)\ q \tag{3.41}$$

$$\Delta q^2(t) = \mathrm{Tr}\big\{\hat{\varrho}(t)\big[\hat{q} - \overline{q(t)}\,\big]^2\big\} = \int \mathrm{d}p\ \mathrm{d}q\ W(p,q,t)\big[q - \overline{q(t)}\,\big]^2. \tag{3.42}$$

The last expressions, which are formally identical to the classical equations (2.15) and (2.16), have been obtained by introducing the Wigner function $W(p,q,t)$ through the relation $\varrho(q,q,t) = \int \mathrm{d}p\, W(p,q,t)$. For $\gamma \to 0$, the effect of the measurement vanishes. In this case, equation (3.37) becomes the Schrödinger equation for the isolated system and the variance (3.42) reduces to the standard quantum mechanical expression.

Finally, we show that the von Neumann collapse theory is recovered for $\gamma \to \infty$ and $T \to \infty$ with $T\gamma^{-1}$ constant. In this limit, we must identify the shortest time scale τ of the classical system with γ^{-1} and the condition $T\gamma^{-1}$ constant allows the inequality $\hbar/k_{\mathrm{B}}T \ll \Omega^{-1} \ll \tau$ to be satisfied always. At time $t = t' + \tau$, equations (3.36) and (3.37) give

$$\varrho(q_1,q_2,t'+\tau) \simeq \exp\big[-\tfrac{1}{2}\kappa\tau(q_1-q_2)^2\big]\,\varrho(q_1,q_2,t') \tag{3.43}$$

$$\psi_{\xi'}(q,t'+\tau) \simeq \exp\left[-\kappa\tau\left(q - q' - \frac{\xi'}{2\sqrt{\kappa}}\right)^2\right]\exp\left(\frac{\xi'^2\tau}{4}\right)\psi(q,t') \tag{3.44}$$

where $\kappa = 2m\gamma k_{\mathrm{B}}T/\hbar^2$, $\xi' = \xi(t')$ and $q' = q(t')$. For $T\gamma^{-1}$ constant and $\tau = \gamma^{-1} \to 0$, equations (3.43) and (3.44) provide an instantaneous diagonalization of the reduced density matrix and an instantaneous collapse of the wavefunction into the eigenfunction of $\hat{q}$ corresponding to the eigenvalue $q' + \xi'/2\sqrt{\kappa}$. The normalization factor in (3.44) is such that the quantum expectation values at time $t'+\tau$, when averaged over the noise realizations ξ', coincide with the quantum expectation values at time t'. For instance, we have

$$\begin{aligned}\overline{\langle\psi_{\xi'}(t'+\tau)|\hat{q}|\psi_{\xi'}(t'+\tau)\rangle} &= \int \mathrm{d}\xi'\ \sqrt{\frac{\tau}{2\pi}}\ \exp\left(-\frac{\xi'^2\tau}{2}\right)\int \mathrm{d}q\ q\big|\psi_{\xi'}(q,t'+\tau)\big|^2 \\ &= \int \mathrm{d}q\ q|\psi(q,t')|^2\end{aligned} \tag{3.45}$$

which is the result expected on the basis of the von Neumann postulate, namely: in a selective measurement of position at time t', the probability that the state $|\psi(t')\rangle$ collapses into the eigenstate $|q\rangle$ is $|\langle q|\psi(t')\rangle|^2$.

4. Measurement results

In the previous sections, we have seen how the evolution of a system, classical or quantum, is influenced by coupling its coordinate to those of infinitely many linear oscillators. We called this process a measurement of position in agreement with the fact that in a proper limit the von Neumann collapse theory can be recovered from the resulting equations. We now specify how the properties of the measured system can be operatively read by the observer.

From the point of view of the observer the linear oscillators of the meter always have classical features so that their coordinates can be directly taken as pointers of the meter itself. Consider, for example, a pointer whose value $R(t)$ is defined as

$$R(t) = \int \mathrm{d}s \sum_n \zeta_n(t-s) Q_n(s). \tag{4.1}$$

During a single measurement, i.e. in a selective process, $R(t)$ is a stochastic variable. We would like to choose the response functions $\zeta_n(t-s)$ in order that the statistical properties of $R(t)$ over the ensemble of all possible measurements coincide with or, at least, allow us to recover, the non-selective properties of the measured system. For instance, we could ask that the average measurement result

$$\overline{R(t)} = \int \mathrm{d}s \sum_n \zeta_n(t-s) \overline{Q_n(s)} \tag{4.2}$$

and its variance

$$\Delta R^2(t) = \int \mathrm{d}s \sum_n \zeta_n(t-s) \left[\overline{Q_n(s)^2} - \overline{Q_n(s)}^2 \right] \tag{4.3}$$

correspond to the quantities (2.15) and (2.16) in the case of a classical system or those (3.41) and (3.42) in the case a quantum one.

In the classical case, equation (2.4) provides an explicit expression of the oscillator coordinates. By using the property $\Omega^{-1} \ll \tau$, we see that for $\omega_n \sim \Omega$ we can find a period λ^{-1} much shorter than the fastest classical time and much longer than the inverse of the oscillator frequency, so that the average of $Q_n(t)$ in this period coincides with $q(t)$. With the choice $\zeta_n(t) = \lambda \exp(-\lambda t) \delta_{\omega_n - \Omega}$ and using the definition (2.9) we then have

$$\overline{R(t)} = \overline{q(t)} \tag{4.4}$$

and

$$\Delta R^2(t) = \Delta q^2(t) + \ell^2. \tag{4.5}$$

The pointer variance is the sum of the variance $\Delta q^2(t)$ of the measured system and the resolution of the measurement apparatus

$$\ell^2 = \frac{k_B T}{M \Omega^2}. \tag{4.6}$$

The term ℓ^2 represents a systematic error of the measurement and can, in principle, be subtracted.

The above results can be derived in an alternative way. Consider the general definition of the moments

$$\overline{Q_n(t)} = \int \mathrm{d}p \, \mathrm{d}q \, \mathrm{d}P \, \mathrm{d}Q \; W_{\mathrm{tot}}(p, q, P, Q, t) \, Q_n \tag{4.7}$$

$$\overline{Q_n(t)^2} = \int \mathrm{d}p \, \mathrm{d}q \, \mathrm{d}P \, \mathrm{d}Q \; W_{\mathrm{tot}}(p, q, P, Q, t) \, Q_n^2. \tag{4.8}$$

Here, $W_{\mathrm{tot}}(p, q, P, Q, t)$ is the probability density solution of the Liouville equation for the total system with initial conditions

$$W_{\mathrm{tot}}(p, q, P, Q, t') = \delta(p - p')\delta(q - q') \frac{\exp\left(-H_m(P', Q' - q')/k_B T\right)}{\int \mathrm{d}Q' \, \mathrm{d}P' \, \exp\left(-H_m(P', Q' - q')/k_B T\right)}. \tag{4.9}$$

The oscillators with frequency $\omega_n \sim \Omega$ approach the thermal equilibrium around the instantaneous value of the measured coordinate on a time scale much shorter than a characteristic period λ^{-1} with $\Omega^{-1} \ll \lambda^{-1} \ll \tau$. For these oscillators the time average of the moments (4.7) and (4.8) over a period λ^{-1} can be approximated by inserting the following adiabatic expression for the total probability density in the same equations (4.7) and (4.8):

$$W_{\rm tot}(p, q, P, Q, t) \simeq W(p, q, t) \frac{\exp\big(-H_m(P, Q - q(t))/k_{\rm B}T\big)}{\int {\rm d}Q\, {\rm d}P\ \exp\big(-H_m(P, Q - q(t))/k_{\rm B}T\big)} \tag{4.10}$$

where $W(p, q, t)$ is the solution of (2.14). Equations (4.4) and (4.5) are then obtained by evaluating the Gaussian integrals in (4.7) and (4.8).

The last approach also applies formally at the quantum level by interpreting the W's as Wigner functions. The adiabatic approximation (4.10) becomes

$$W_{\rm tot}(p, q, P, Q, t) \simeq W(p, q, t) W_m(P, Q - q(t)) \tag{4.11}$$

where $W(p, q, t)$ is the Wigner function associated with $\hat{\varrho}(t)$ and

$$W_m(P, Q - q(t)) = \prod_n \frac{\tanh(\hbar\omega_n/2k_{\rm B}T)}{\pi\hbar} \exp\left\{- \tanh\left(\frac{\hbar\omega_n}{2k_{\rm B}T}\right) \times \left[\frac{P_n^2}{\hbar M\omega_n} + \frac{M\omega_n}{\hbar}\big(Q_n - q(t)\big)^2\right]\right\} \tag{4.12}$$

is the Wigner function associated with the density matrix (3.3), with $Q_{\rm eq} = q(t)$. Here, $q(t)$ is the quantum expectation value of $\hat{q}$ in the state $|\psi_{[\xi]}(t)\rangle$. Due to the condition $k_{\rm B}T \gg \hbar\Omega$, the above expression reduces to the classical distribution

$$W_m(P, Q - q(t)) = \prod_n \frac{\omega_n}{2\pi k_{\rm B}T} \exp\left\{-\left[\frac{P_n^2}{2M} + \frac{M\omega_n^2}{2}\big(Q_n - q(t)\big)^2\right]\frac{1}{k_{\rm B}T}\right\} \tag{4.13}$$

so that equations (4.4) and (4.5) still hold with the same resolution ℓ of the measurement apparatus.

5. The $\hbar \to 0$ limit: definition of coherent states

The dynamics of a closed quantum system reduces to the classical dynamics in the $\hbar \to 0$ limit only if the system is prepared in appropriate states. The coherent states defined as the ground state of the displaced harmonic oscillator

$$|p'q'\rangle = {\rm e}^{-({\rm i}/\hbar)q'\hat{p}}\ {\rm e}^{({\rm i}/\hbar)p'\hat{q}}|\phi_0\rangle \tag{5.1}$$

where $|\phi_0\rangle$ is the ground state of the undisplaced oscillator, are a well known example [27]. These states provide a convenient representation for studying the $\hbar \to 0$ limit regardless of the nature of the Hamiltonian which may not preserve their form [28].

In the case of the measurement model discussed here, it is possible to find states which are localized and stationary in the co-moving frame of the measured system and play the role of the ground state $|\phi_0\rangle$ in equation (5.1). A first example of these states was given in [15, 18] for a free particle evolving according to the dissipationless equation (A.6). A generalization valid in the case of a harmonic oscillator described by (3.37) has recently been provided in [19]. Here, we derive the expression of the coherent states for a general linear system with constant proper frequency undergoing the selective measurement process of (3.37). Then, we discuss the recovering of the classical limit in selective and non-selective measurement processes on arbitrary systems which are prepared in such states.

During a selective measurement, the quantum system is described by a state $|\psi_{[\xi]}(t)\rangle$ which evolves in a specified rest-frame according to (3.37). In analogy with (5.1), we seek solutions of (3.37) of the form

$$|\psi_{[\xi]}(t)\rangle = \mathrm{e}^{-(\mathrm{i}/\hbar)q(t)\hat{p}}\ \mathrm{e}^{(\mathrm{i}/\hbar)p(t)\hat{q}}\ \mathrm{e}^{-(\mathrm{i}/\hbar)\varphi(t)}|\phi\rangle \tag{5.2}$$

where $p(t) = \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle$ and $q(t) = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$. The co-moving state $|\phi\rangle$ is assumed constant so that the solutions (5.2) depend on the noise $\xi(t)$ only through the expectation values $p(t)$ and $q(t)$ and the action $\varphi(t)$. By inverting the transformation (5.2) and imposing the requirement that the change of $|\phi\rangle$ in a time dt vanishes, we obtain

$$\mathrm{e}^{(\mathrm{i}/\hbar)[\varphi(t)+\mathrm{d}\varphi(t)]}\ \mathrm{e}^{-(\mathrm{i}/\hbar)[p(t)+\mathrm{d}p(t)]\hat{q}}\ \mathrm{e}^{(\mathrm{i}/\hbar)[q(t)+\mathrm{d}q(t)]\hat{p}}\big[|\psi_{[\xi]}(t)\rangle + \mathrm{d}|\psi_{[\xi]}(t)\rangle\big] - |\phi\rangle = 0. \tag{5.3}$$

The differential $\mathrm{d}|\psi_{[\xi]}(t)\rangle$ is given by equation (3.37). The same equation (3.37) allows the evaluation of

$$\mathrm{d}p(t) = -\big[\gamma p(t) + \langle\psi_{[\xi]}(t)|\partial_q\hat{V}(\hat{q},t)|\psi_{[\xi]}(t)\rangle\big]\,\mathrm{d}t + 2\sqrt{\kappa}\sigma_{pq}^2\,\mathrm{d}w_\xi(t) \tag{5.4}$$

and

$$\mathrm{d}q(t) = \frac{p(t)}{m}\,\mathrm{d}t + \left(2\sqrt{\kappa}\sigma_q^2 - \frac{\gamma}{2\sqrt{\kappa}}\right)\mathrm{d}w_\xi(t) \tag{5.5}$$

where

$$\sigma_q^2 = \langle\psi_{[\xi]}(t)|\hat{q}^2|\psi_{[\xi]}(t)\rangle - \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle^2 = \langle\phi|\hat{q}^2|\phi\rangle \tag{5.6}$$

and

$$\begin{aligned}\sigma_{pq}^2 &= \tfrac{1}{2}\langle\psi_{[\xi]}(t)|\hat{p}\hat{q}+\hat{q}\hat{p}|\psi_{[\xi]}(t)\rangle - \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle\langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle \\ &= \tfrac{1}{2}\langle\phi|\hat{p}\hat{q}+\hat{q}\hat{p}|\phi\rangle\end{aligned} \tag{5.7}$$

are the constant variances associated with the states (5.2) and $\kappa = 2m\gamma k_\mathrm{B}T/\hbar^2$. The expectation value of the force operator which appears in (5.4) can be expressed in terms of the co-moving state $|\phi\rangle$ by a translation $\hat{q} \to \hat{q} + q(t)$:

$$\langle\psi_{[\xi]}(t)|\partial_q\hat{V}(\hat{q},t)|\psi_{[\xi]}(t)\rangle = \langle\phi|\partial_q\hat{V}(\hat{q}+q(t),t)|\phi\rangle. \tag{5.8}$$

Finally, we write the differential of the action $\varphi(t)$ in terms of two coefficients $\mu(t)$ and $\nu(t)$ to be determined later

$$\mathrm{d}\varphi(t) = \mu(t)\,\mathrm{d}t + \nu(t)\,\mathrm{d}w_\xi(t). \tag{5.9}$$

By expanding the exponentials and using the Ito rule, equation (5.3) can be rewritten as

$$\big[\hat{1} + \hat{F}\,\mathrm{d}w_\xi(t) + \hat{G}\,\mathrm{d}t\big]|\phi\rangle - |\phi\rangle = 0 \tag{5.10}$$

which is equivalent to $\hat{F}|\phi\rangle = 0$ and $\hat{G}|\phi\rangle = 0$. In general, the operators $\hat{F}$ and $\hat{G}$ will depend on time through the expectation values $p(t)$ and $q(t)$ and the action $\varphi(t)$, so that these equations cannot be satisfied with a constant $|\phi\rangle$. However, we can try to make $\hat{F}$ and $\hat{G}$ time independent with a proper choice of the coefficients $\mu(t)$ and $\nu(t)$. In the case of $\hat{F}$, we have

$$\hat{F} = \sqrt{\kappa}\left[\left(1 - \frac{2\mathrm{i}}{\hbar}\sigma_{pq}^2\right)\hat{q} + \frac{2\mathrm{i}}{\hbar}\sigma_q^2\hat{p}\right] + \frac{\mathrm{i}}{\hbar}\left[\nu(t) + p(t)\left(2\sqrt{\kappa}\sigma_q^2 - \frac{\gamma}{2\sqrt{\kappa}}\right)\right] \tag{5.11}$$

and this becomes time independent with the choice

$$\nu(t) = -p(t)\left(2\sqrt{\kappa}\sigma_q^2 - \frac{\gamma}{2\sqrt{\kappa}}\right). \tag{5.12}$$

The corresponding equation $\hat{F}|\phi\rangle = 0$ has the unique normalized solution

$$\phi(q) = \langle q|\phi\rangle = \left(2\pi\sigma_q^2\right)^{-1/4} \exp\left(-\frac{1 - (2\mathrm{i}/\hbar)\sigma_{pq}^2}{4\sigma_q^2}\, q^2\right). \tag{5.13}$$

For $\nu(t)$ given by (5.12), the operator $\hat{G}$ is

$$\hat{G} = -\frac{\mathrm{i}}{\hbar}\left[\frac{\hat{p}^2}{2m} + \hat{V}(\hat{q} + q(t), t) - \langle\phi|\partial_q \hat{V}(\hat{q} + q(t), t)|\phi\rangle\hat{q} - \mu(t) - \frac{p(t)^2}{2m} + \frac{\gamma}{2} p(t)q(t) \right.$$
$$\left. + \frac{\gamma}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) - 2\kappa\sigma_q^2\sigma_{pq}^2\right] - \kappa\left(\hat{q}^2 - \sigma_q^2\right) + \frac{1}{2}\hat{F}^2. \tag{5.14}$$

This can be made time independent with a proper choice of $\mu(t)$ only for linear systems. Assuming $V(q,t) = v_0(t) + v_1(t)q + \frac{1}{2}m\omega_0^2 q^2$ with ω_0 constant, we have

$$\hat{V}(\hat{q} + q(t), t) - \langle\phi|\partial_q \hat{V}(\hat{q} + q(t), t)|\phi\rangle\hat{q} = V(q(t), t) + \tfrac{1}{2}m\omega_0^2\hat{q}^2 \tag{5.15}$$

so that by choosing

$$\mu(t) = \epsilon - \frac{p(t)^2}{2m} + V(q(t), t) + \frac{\gamma}{2} p(t)q(t) \tag{5.16}$$

the equation $\hat{G}|\phi\rangle = 0$ becomes

$$\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{q}^2 + \frac{\gamma}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) - 2\kappa\sigma_q^2\sigma_{pq}^2 - \mathrm{i}\hbar\kappa\left(\hat{q}^2 - \sigma_q^2\right)\right]|\phi\rangle = \epsilon|\phi\rangle. \tag{5.17}$$

In the position representation and using $\phi(q)$ given by (5.13), equation (5.17) is equivalent to the following two complex equations:

$$-\frac{\hbar^2}{2m}\left(\frac{1 - (2\mathrm{i}/\hbar)\sigma_{pq}^2}{2\sigma_q^2}\right)^2 + \frac{m\omega_0^2}{2} + \frac{\mathrm{i}\hbar\gamma}{2}\,\frac{1 - (2\mathrm{i}/\hbar)\sigma_{pq}^2}{\sigma_q^2} - \mathrm{i}\hbar\kappa = 0 \tag{5.18}$$

$$\frac{\hbar^2}{2m}\,\frac{1 - (2\mathrm{i}/\hbar)\sigma_{pq}^2}{2\sigma_q^2} - \frac{\mathrm{i}\hbar\gamma}{2} - 2\kappa\sigma_q^2\sigma_{pq}^2 + \mathrm{i}\hbar\kappa\sigma_q^2 = \epsilon. \tag{5.19}$$

Equation (5.18) gives two conditions for the determination of σ_q^2 and σ_{pq}^2. The constant ϵ can be then evaluated from the real part of (5.19), the imaginary part being an identity. The solutions are

$$\sigma_q^2 = \sqrt{\frac{\gamma^2 - \omega_0^2 + \sqrt{(\gamma^2 - \omega_0^2)^2 + (2\hbar\kappa/m)^2}}{8\kappa^2}} \tag{5.20}$$

$$\sigma_{pq}^2 = \sqrt{m^2(\gamma^2 - \omega_0^2)\sigma_q^4 + \tfrac{1}{4}\hbar^2} - m\gamma\sigma_q^2 \tag{5.21}$$

and

$$\epsilon = \frac{\hbar^2}{4m\sigma_q^2} - 2\kappa\sigma_q^2\sigma_{pq}^2. \tag{5.22}$$

The variances σ_q^2 and σ_{pq}^2 are always real and positive except for $k_\mathrm{B}T/\hbar\omega_0 \ll 1$ which is, however, outside the range of validity of (3.37). According to the choices (5.12) and (5.16), we finally have

$$\mathrm{d}\varphi(t) = \left[\epsilon + \frac{p(t)^2}{2m} + V(q(t), t) + \frac{\gamma}{2} p(t)q(t)\right]\mathrm{d}t - p(t)\,\mathrm{d}q(t). \tag{5.23}$$

This allows an interpretation of ϵ in terms of a zero-point energy which adds to the classical renormalized Hamiltonian $H(p(t), q(t), t) + \frac{1}{2}\gamma p(t)q(t)$.

In the $\gamma \to 0$ limit, we have $\sigma_q^2 = \hbar/2m\omega_0$, $\sigma_{pq}^2 = 0$ and $\epsilon = \hbar\omega_0/2$. The stationary state of (5.13) becomes the ground state $|\phi_0\rangle$ of an unmeasured harmonic oscillator with frequency ω_0. In analogy with (5.1), the coherent states in presence of a continuous measurement are then defined as

$$|p'q'\rangle = \mathrm{e}^{-(\mathrm{i}/\hbar)q'\hat{p}}\,\mathrm{e}^{(\mathrm{i}/\hbar)p'\hat{q}}|\phi\rangle \tag{5.24}$$

which, in the position representation, becomes

$$\langle q|p'q'\rangle = \left(2\pi\sigma_q^2\right)^{-1/4}\exp\left[-\frac{1-(2\mathrm{i}/\hbar)\sigma_{pq}^2}{4\sigma_q^2}(q-q')^2 + \frac{\mathrm{i}}{\hbar}p'(q-q')\right] \tag{5.25}$$

with σ_q^2 and σ_{pq}^2 given by (5.20) and (5.21), respectively. The states $|p'q'\rangle$ of equation (5.24) have the same properties of the usual coherent states [27]. In particular, they form an overcomplete basis with the completeness relationship

$$\int \frac{\mathrm{d}p'\,\mathrm{d}q'}{2\pi\hbar}\,|p'q'\rangle\langle p'q'| = \hat{1} \tag{5.26}$$

and overlaps

$$\langle p'q'|p''q''\rangle = \exp\left[-C_{p'-p''q'-q''} + \frac{\mathrm{i}}{\hbar}\,\frac{p'+p''}{2}(q'-q'')\right] \tag{5.27}$$

where

$$C_{p'-p''q'-q''} = \frac{\sigma_q^2}{2\hbar^2}\left[(p'-p'') - \frac{\sigma_{pq}^2}{\sigma_q^2}(q'-q'')\right]^2 + \frac{1}{8\sigma_q^2}(q'-q'')^2. \tag{5.28}$$

Suppose that a quantum system, not necessarily a linear one, is prepared at time t in the coherent state $|p(t)q(t)\rangle$. To leading order in $\hbar$, we have

$$\sigma_q^2 = \sqrt{\frac{\hbar^3}{8m^2\gamma k_{\mathrm{B}}T}} \tag{5.29}$$

and

$$\sigma_{pq}^2 = \tfrac{1}{2}\hbar. \tag{5.30}$$

We also have

$$\begin{aligned}\sigma_p^2 &= \langle p(t)q(t)|\hat{p}^2|p(t)q(t)\rangle - \langle p(t)q(t)|\hat{p}|p(t)q(t)\rangle^2 = \langle\phi|\hat{p}^2|\phi\rangle \\ &= \frac{\frac{1}{4}\hbar^2 + \sigma_{pq}^4}{\sigma_q^2} = \frac{\hbar^2}{2\sigma_q^2} = \sqrt{2m^2\hbar\gamma k_{\mathrm{B}}T}.\end{aligned} \tag{5.31}$$

Note that these expressions are independent of ω_0†. Since σ_p^2, σ_q^2 and σ_{pq}^2 vanish for $\hbar \to 0$, in this limit the expectation values $p(t) = \langle p(t)q(t)|\hat{p}|p(t)q(t)\rangle$ and $q(t) =$

† Equations (5.29) and (5.30) are also the leading-order terms of equations (5.6) and (5.7) with respect to the small parameter $\hbar\gamma/k_{\mathrm{B}}T$. They correspond to the particular case $V(x) = 0$ and $b = 0$ given in [19, equation (3.27)] In that paper the definitions of γ and of the Wiener process are different from ours and this implies different numerical factors which, however, cancel out in these simplified expressions of σ_p^2, σ_{pq}^2 and σ_q^2.

$\langle p(t)q(t)|\hat{q}|p(t)q(t)\rangle$ can be interpreted as classical phase-space coordinates. In a selective measurement, their change is given by equations (5.4) and (5.5):

$$\mathrm{d}p(t) = -\left[\gamma p(t) + \langle p(t)q(t)|\partial_q \hat{V}(\hat{q},t)|p(t)q(t)\rangle\right]\mathrm{d}t + \sqrt{2m\gamma k_{\mathrm{B}}T}\,\mathrm{d}w_\xi(t) \tag{5.32}$$

$$\mathrm{d}q(t) = \frac{p(t)}{m}\,\mathrm{d}t + \left(\sqrt{\frac{\hbar}{m}} - \sqrt{\frac{\hbar^2\gamma}{8mk_{\mathrm{B}}T}}\right)\mathrm{d}w_\xi(t). \tag{5.33}$$

For $\hbar \to 0$, the expectation value $\langle p(t)q(t)|\partial_q \hat{V}(\hat{q},t)|p(t)q(t)\rangle$ can be replaced with $\partial_q V(q(t),t)$ and we recover the classical Langevin equations (2.12) and (2.13).

In the case of a non-selective measurement, the $\hbar \to 0$ limit is properly discussed in terms of the Wigner function related to the reduced density matrix operator through the transformation

$$W(p,q,t) = \frac{1}{2\pi\hbar}\int \mathrm{d}z\ \exp\left(\frac{\mathrm{i}}{\hbar}pz\right)\langle q - \tfrac{1}{2}z|\hat{\varrho}(t)|q + \tfrac{1}{2}z\rangle. \tag{5.34}$$

Suppose that at time t the measured system is described by the density matrix obtained by averaging the state $|p(t)q(t)\rangle$ over all noise realizations, i.e. all possible values of $p(t)$ and $q(t)$ specified by a certain distribution function such that $\mathrm{Tr}\,\hat{\varrho}(t) = 1$:

$$\hat{\rho}(t) = \overline{|p(t)q(t)\rangle\langle p(t)q(t)|} = \int \mathrm{d}p'\,\mathrm{d}q'\,\overline{\delta(p'-p(t))\delta(q'-q(t))}\ |p'q'\rangle\langle p'q'|. \tag{5.35}$$

The corresponding Wigner function is

$$W(p,q,t) = \int \mathrm{d}p'\,\mathrm{d}q'\,\overline{\delta(p'-p(t))\delta(q'-q(t))}\ W_{p'q'}(p,q) \tag{5.36}$$

where

$$\begin{aligned} W_{p'q'}(p,q) &= \frac{1}{2\pi\hbar}\int \mathrm{d}z\ \exp\left(\frac{\mathrm{i}}{\hbar}pz\right)\langle q - \tfrac{1}{2}z|p'q'\rangle\langle p'q'|q + \tfrac{1}{2}z\rangle \\ &= \frac{1}{\pi\hbar}\exp\left\{-\frac{2\sigma_q^2}{\hbar^2}\left[(p-p') - \frac{\sigma_{pq}^2}{\sigma_q^2}(q-q')\right]^2 - \frac{1}{2\sigma_q^2}(q-q')^2\right\}. \end{aligned} \tag{5.37}$$

Since

$$\lim_{\hbar\to 0} W_{p'q'}(p,q) = \delta(p-p')\delta(q-q') \tag{5.38}$$

in the $\hbar \to 0$ limit $W(p,q,t)$ reduces to the classical probability density $\overline{\delta(p-p(t))\delta(q-q(t))}$ obtained by averaging the sharp density $\delta(p-p(t))\delta(q-q(t))$ over all acceptable phase-space points $p(t)$, $q(t)$. Finally, the equation of motion of the Wigner function, obtained from equation (3.36) with standard manipulations [29], is

$$\begin{aligned} \partial_t W(p,q,t) = \Bigg[&-\frac{p}{m}\partial_q + \sum_{n=0}^{\infty}\left(\frac{\hbar}{2\mathrm{i}}\right)^{2n}\frac{1}{(2n+1)!}\partial_q^{2n+1}V(q,t)\partial_p^{2n+1} \\ &+ \partial_p\left(\gamma p + m\gamma k_{\mathrm{B}}T\partial_p\right) + \frac{\hbar^2\gamma}{16mk_{\mathrm{B}}T}\partial_q^2\Bigg]W(p,q,t) \end{aligned} \tag{5.39}$$

so that, in the $\hbar \to 0$ limit, the change of $W(p,q,t)$ coincides with that prescribed by the classical Fokker–Plank equation (2.14).

6. Measurements on macroscopic systems

One of the principal drawbacks of the von Neumann measurement theory is the impossibility of predicting a quantum-to-classical transition in the macroscopic limit, unless the state $|\psi(t')\rangle$ of the system at the beginning of the measurement is one of the coherent states (5.1). On the other hand, when the size of the system is sufficiently large, i.e. in the formal $\hbar \to 0$ limit, we must always recover the result of a classical measurement. It is now clearly established that the entanglement of the measured system with the infinitely many degrees of freedom of the measurement apparatus can provide superselection rules which avoid paradoxical quantum features at macroscopic level [4, 5].

Concerning the measurement model discussed here, the existence of a superselection rule of this kind can be demonstrated in a general way in the case of linear systems. Halliwell and Zoupas have shown, in a statistical sense first [19] and with a more direct approach but in the free-particle case and dissipationless limit later [30], that the solutions of equation (3.37) converge to a coherent state characterized by time-dependent parameters $p(t)$ and $q(t)$ which are the expectation values of $\hat{p}$ and $\hat{q}$ in the state itself. Here, we generalize the result of [30] by showing that the solutions of equation (3.37) with potential

$$V = v_0(t) + v_1(t)q + \tfrac{1}{2}m\omega_0^2 q^2 \tag{6.1}$$

in the long-time limit become of the form

$$|\psi_{[\xi]}(t)\rangle = \exp\left[-\frac{\mathrm{i}}{\hbar}\varphi(t)\right] \, |p(t)q(t)\rangle \tag{6.2}$$

where $p(t) = \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle$, $q(t) = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$ and $\varphi(t)$ and $|p(t)q(t)\rangle$ are given by equations (5.23) and (5.24), respectively.

The Green function corresponding to the nonlinear equation (3.37)

$$\begin{aligned} G_{[\xi]}(q,t;q',t') = \int \mathrm{d}[p]\,\mathrm{d}[q]_{q',t'}^{q,t}\ \exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}s\ \Bigg[p\dot{q} - \frac{p^2}{2m} - V - \gamma p q + \gamma p\langle\hat{q}\rangle \\ - \frac{\gamma}{2}\langle\hat{p}\rangle\langle\hat{q}\rangle + \mathrm{i}\hbar\kappa(q-\langle\hat{q}\rangle)^2 - \mathrm{i}\hbar\sqrt{\kappa}(q-\langle\hat{q}\rangle)\xi \\ + \frac{\gamma}{2\sqrt{\kappa}}(p-\langle\hat{p}\rangle)\xi\Bigg]\Bigg\} \end{aligned} \tag{6.3}$$

depends functionally on the state $|\psi_{[\xi]}(t)\rangle$ through the expectation values $\langle\hat{p}\rangle = \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle$ and $\langle\hat{q}\rangle = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$. If we suppose, for the moment, that these functions and the noise ξ are given, for V of the form (6.1) the Green function (6.3) is that of a linear system with Lagrangian

$$L(q,\dot{q},t) = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2 q^2 - m\gamma q\dot{q} + f(t)q + g(t)\dot{q} + h(t) \tag{6.4}$$

where

$$\omega^2 = \omega_0^2 - \gamma^2 - \frac{2\mathrm{i}\hbar\kappa}{m} \tag{6.5}$$

and $f(t)$, $g(t)$ and $h(t)$ are given in terms of $\langle\hat{p}\rangle$, $\langle\hat{q}\rangle$, ξ, v_0 and v_1. By performing the Gaussian functional integrals in equation (6.3), we obtain

$$\begin{aligned} G_{[\xi]}(q,t;q',t') = n(t,t')\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\Bigg[g(t)q - g(t')q' - \frac{1}{2}m\gamma q^2 + \frac{1}{2}m\gamma q'^2 \\ + S_{\mathrm{cl}}(q,t;q',t')\Bigg]\Bigg\} \end{aligned} \tag{6.6}$$

where $n(t, t')$ does not depend on the spatial variables and

$$S_{\mathrm{cl}}(q, t; q', t') = \frac{1}{\sin[\omega(t-t')]}\left\{\frac{m\omega}{2}\cos[\omega(t-t')]\left(q^2+q'^2\right) - m\omega q q'\right.$$
$$+ q\int_{t'}^{t}\mathrm{d}s\ [f(s)-\dot{g}(s)]\sin[\omega(s-t')]$$
$$+ q'\int_{t'}^{t}\mathrm{d}s\ [f(s)-\dot{g}(s)]\sin[\omega(t-s)]$$
$$-\frac{1}{m\omega}\int_{t'}^{t}\mathrm{d}s\ [f(s)-\dot{g}(s)]\sin[\omega(t-s)]$$
$$\left.\times\int_{t'}^{s}\mathrm{d}u\ [f(u)-\dot{g}(u)]\sin[\omega(u-t)]\right\} \tag{6.7}$$

is the classical action of a driven harmonic oscillator of mass m, frequency ω and external force $f-\dot{g}$, evaluated with boundary conditions $q(t') = q'$ and $q(t) = q$ [31]. The frequency ω is complex with real and imaginary parts given by

$$\mathrm{Re}(\omega) = \pm\frac{\hbar}{2m}\sqrt{\frac{8\kappa^2}{\gamma^2-\omega_0^2+\sqrt{(\gamma^2-\omega_0^2)^2+(2\hbar\kappa/m)^2}}} = \pm\frac{\hbar}{2m\sigma_q^2} \tag{6.8}$$

$$\mathrm{Im}(\omega) = \mp\sqrt{\gamma^2-\omega_0^2+\left(\frac{\hbar}{2m\sigma_q^2}\right)^2}. \tag{6.9}$$

For $(t-t')|\,\mathrm{Im}(\omega)| \gg 1$, the coefficient of the qq' term in the action (6.7) vanishes while the coefficient of $(q^2+q'^2)$ becomes $\pm\mathrm{i}m\omega/2$. In the long-time limit, therefore, the propagator (6.6) becomes independent of the initial conditions and the solutions of equation (3.37) can be written as

$$\psi_{[\xi]}(q,t) = \int\mathrm{d}q'\ G_{[\xi]}(q,t;q',t')\ \psi(q',t')$$
$$= \exp\left[-\frac{m}{2\hbar}\left(\pm\omega+\mathrm{i}\gamma\right)q^2+\alpha(t)q+\beta(t)\right]$$
$$= \exp\left[-\frac{1-(2\mathrm{i}/\hbar)\sigma_{pq}^2}{4\sigma_q^2}\ q^2+\alpha(t)q+\beta(t)\right] \tag{6.10}$$

where we used

$$-\frac{\hbar\,\mathrm{Im}(\omega)}{2\,\mathrm{Re}(\omega)} \mp \frac{\hbar\gamma}{2\,\mathrm{Re}(\omega)} = \sqrt{m^2(\gamma^2-\omega_0^2)\sigma_q^4+\frac{\hbar^2}{4}} - m\gamma\sigma_q^2 = \sigma_{pq}^2. \tag{6.11}$$

The complex functions $\alpha(t)$ and $\beta(t)$ are to be determined. First of all, the normalization of the wavefunction (6.10) implies that

$$1 = (2\pi\sigma_q^2)^{1/2}\exp\left\{2\,\mathrm{Re}[\beta(t)]+2\sigma_q^2\,\mathrm{Re}[\alpha(t)]^2\right\}. \tag{6.12}$$

Then, we can impose two self-consistency conditions involving the expectation values of $\hat{p}$ and $\hat{q}$ in the state (6.10)

$$p(t) = \langle\psi_{[\xi]}(t)|\hat{p}|\psi_{[\xi]}(t)\rangle = \hbar\,\mathrm{Im}[\alpha(t)] + 2\sigma_{pq}^2\,\mathrm{Re}[\alpha(t)] \tag{6.13}$$

and

$$q(t) = \langle \psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle = 2\sigma_q^2 \,\mathrm{Re}[\alpha(t)]. \tag{6.14}$$

By using the expressions so obtained for $\mathrm{Re}[\alpha(t)]$, $\mathrm{Im}[\alpha(t)]$ and $\mathrm{Re}[\beta(t)]$, the wavefunction (6.10) can be rewritten as

$$\psi_{[\xi]}(q,t) = (2\pi\sigma_q^2)^{-1/4} \exp\left\{-\frac{1-(2\mathrm{i}/\hbar)\sigma_{pq}^2}{4\sigma_q^2}\,[q-q(t)]^2 + \frac{\mathrm{i}}{\hbar}p(t)[q-q(t)] - \frac{\mathrm{i}}{\hbar}\varphi(t)\right\} \tag{6.15}$$

where

$$\varphi(t) = -\hbar\,\mathrm{Im}[\beta(t)] - p(t)q(t) + \frac{\sigma_{pq}^2}{2\sigma_q^2}q(t)^2. \tag{6.16}$$

The action $\varphi(t)$ evolves according to an equation obtained by imposing that the change of $\psi_{[\xi]}(q,t)$ in a time $\mathrm{d}t$ is given by (3.37). Since the wavefunction (6.15) is of the form (5.2), the differential $\mathrm{d}\varphi(t)$ is given by (5.23). This completes the convergence proof.

To the leading order in $\hbar$, the characteristic time which determines the convergence of $\psi_{[\xi]}(q,t)$ to the wavefunction (6.15) is given by

$$\frac{1}{|\,\mathrm{Im}(\omega)|} = \sqrt{\frac{\hbar}{\gamma k_\mathrm{B} T}}. \tag{6.17}$$

The convergence becomes infinitely fast for $\hbar \to 0$. On the base of this result and of the properties of the coherent states discussed in section 5, we can conclude that during a measurement, selective or non-selective, the $\hbar \to 0$ limit does exist at any time $t > t'$ even if it does not exist at $t = t'$. As an example of this behaviour, in appendix B we explicitly evaluate the $\hbar \to 0$ limit in the case of non-selective measurements on a free-particle cat state. The discontinuity at $t = t'$ is, of course, an artifact of the instantaneous correlation assumed through equation (3.3) between the measured system and the measurement apparatus and it would disappear in a more physical approach in which such correlation is established in a finite time.

In the case of nonlinear systems, terms higher than quadratic appear in the potential of the Lagrangian (6.4) so that the convergence proof given for linear systems does not apply. However, due to the linearity of the interaction with the infinitely many oscillators of the measurement apparatus, the leading $\hbar$ term of this potential, i.e.

$$-\mathrm{i}\hbar\kappa q^2 = -\mathrm{i}\frac{2m\gamma k_\mathrm{B} T}{\hbar}q^2 \tag{6.18}$$

is a quadratic one with complex frequency $\sqrt{-4\mathrm{i}\gamma k_\mathrm{B} T/\hbar}$. As a consequence, in the $\hbar \to 0$ limit the state of the system acquires the form (6.15) with σ_q and σ_{pq} given by equations (5.29) and (5.30). The recovery of classical behaviour in the macroscopic limit is, therefore, obtained independently of the nature of the measured system. Numerical examples of this result can be found in [32–34] and experimental evidence has recently been reported in [35].

After the completion of this paper, we became aware of a preprint by Strunz and Percival [36] in which the authors discuss the semiclassical behaviour of open quantum systems described by a general Lindblad master equation.

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Appendix A. Positivity and Markovian evolution of $\hat{\varrho}$

Violations of the positivity of $\hat{\varrho}(t)$ may arise due to inappropriate approximations of the exact influence functional (3.12). Examples of exact solution of $\hat{\varrho}(t)$ with no positivity violations have been given in [37] in the case of a harmonic oscillator.

In the framework of a theory of measurement processes, the requirement that the meter has classical behaviour with respect to the observer imposes $\hbar\Omega \ll k_B T$, if the frequency distribution (2.10) is assumed. In this case, the fluctuation kernel

$$\begin{aligned}\Lambda(s-u) &= \frac{m\gamma}{\pi\hbar}\int_0^\Omega d\omega\, \omega \coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos[\omega(s-u)] \\ &= \frac{k_B T}{\hbar^2}\Gamma(s-u) - \frac{1}{12k_B T}\ddot{\Gamma}(s-u) + \cdots \end{aligned} \tag{A.1}$$

has a high-temperature expansion whose leading term $\Gamma k_B T/\hbar^2$ is proportional to the dissipation kernel

$$\Gamma(t-s) \simeq 2m\gamma\delta(t-s) \tag{A.2}$$

which is Markovian for $\Omega^{-1} \ll \tau$, τ being the fastest time scale of the classical motion. When these approximations are made in (3.12), so that equation (3.15) and the corresponding master equation (3.18) are obtained, violations of positivity of $\hat{\varrho}(t)$ may occur as in the example pointed out in [38]. However, this happens on a time scale shorter than $\hbar/k_B T$, i.e. outside the range of validity $\tau \gg \Omega^{-1} \gg \hbar/k_B T$ of equation (3.18) [37]. In this range, the substantial positivity of $\hat{\varrho}(t)$ can be made apparent by selecting appropriate dominant terms.

Equation (3.15) shows that the dissipation term is negligible in comparison with the fluctuation term when

$$\frac{m\gamma k_B T q_\Delta^2}{\hbar^2} \gg \frac{\gamma q_\Delta p_\Sigma}{2\hbar} \tag{A.3}$$

where $q_\Delta = q_1 - q_2$ and $p_\Sigma = p_1 + p_2$. The functions q_Δ and p_Σ may assume any value according to the functional measure (3.11). However, close to the dominant classical path we have $p_\Sigma \lesssim 2mq_\Delta/\tau$ and the condition (A.3) can be restated as $k_B T/\hbar \gg \tau^{-1}$. Therefore, in the working range $\tau \gg \Omega^{-1} \gg \hbar/k_B T$ dissipation can be neglected with respect to fluctuation and equation (3.15) becomes

$$Z[q_1, q_2] = \int_{t'}^{t} ds\, \frac{m\gamma k_B T}{\hbar^2}\big[q_1(s) - q_2(s)\big]^2. \tag{A.4}$$

Correspondingly, the non-selective measurement processes are described by the master equation

$$\frac{d}{dt}\hat{\varrho}(t) = -\frac{i}{\hbar}\big[\hat{H}(\hat{p},\hat{q},t), \hat{\varrho}(t)\big] - \frac{m\gamma k_B T}{\hbar^2}\big[\hat{q},\big[\hat{q},\hat{\varrho}(t)\big]\big]. \tag{A.5}$$

This is a Markovian evolution of Lindblad class and therefore (completely) positive [26]. The associated selective processes are described in terms of a single state satisfying the stochastic Schrödinger equation

$$\mathrm{d}|\psi_{[\xi]}(t)\rangle = -\frac{\mathrm{i}}{\hbar}\hat{H}(\hat{p},\hat{q},t)|\psi_{[\xi]}(t)\rangle\,\mathrm{d}t - \frac{m\gamma k_{\mathrm{B}}T}{\hbar^2}\big[\hat{q}-q(t)\big]^2|\psi_{[\xi]}(t)\rangle\,\mathrm{d}t + \sqrt{\frac{2m\gamma k_{\mathrm{B}}T}{\hbar^2}}\big[\hat{q}-q(t)\big]|\psi_{[\xi]}(t)\rangle\,\mathrm{d}w_\xi(t) \tag{A.6}$$

with $q(t) = \langle\psi_{[\xi]}(t)|\hat{q}|\psi_{[\xi]}(t)\rangle$. The general results of [9] are recovered by setting $2m\gamma k_{\mathrm{B}}T/\hbar^2 = \kappa$.

At the classical level, the Fokker–Plank equation (2.14) and the Langevin equations (2.12) and (2.13) are consistent with the fluctuation–dissipation theorem [23]. Equations (A.5) and (A.6), in which dissipation is neglected, are therefore not appropriate for recovering the classical limit. New quantum equations are to be introduced which include dissipation and, at the same time, guarantee the positivity of $\hat{\varrho}(t)$. As shown in section 3, this is accomplished by rewriting equation (3.15) in the equivalent form (3.21) and neglecting the B terms with respect to the A ones on the basis of the working condition $\hbar/k_{\mathrm{B}}T \ll \tau$. Dissipation is still contained in the remaining influence functional which gives rise to the master equation (3.36) of Lindblad class and to the corresponding stochastic Schrödinger equation (3.37). These equations provide the correct classical limit as shown in section 5.

We conclude with some remarks about the possibility pointed out in [39, 40] of obtaining a master equation of Lindblad class by taking into account the next to leading term in (A.1). In this case, equation (3.15) would become

$$Z[p_1,q_1,p_2,q_2] = \int_{t'}^{t}\mathrm{d}s\,\left\{\frac{m\gamma k_{\mathrm{B}}T}{\hbar^2}\big[q_1(s)-q_2(s)\big]^2 + \frac{\mathrm{i}\gamma}{2\hbar}\big[q_1(s)-q_2(s)\big]\big[p_1(s)+p_2(s)\big]\right\} - \int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}u\;\frac{1}{12k_{\mathrm{B}}T}\big[q_1(s)-q_2(s)\big]\ddot{\Gamma}(s-u)\big[q_1(u)-q_2(u)\big] + \cdots. \tag{A.7}$$

The new term can be more easily analysed after integration by parts. By setting $q_\Delta = q_1 - q_2$, we have

$$\int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}u\;q_\Delta(s)\ddot{\Gamma}(s-u)q_\Delta(u) = \frac{1}{2}\int_{t'}^{t}\mathrm{d}s\int_{t'}^{t}\mathrm{d}u\;\ddot{q}_\Delta(s)\Gamma(s-u)q_\Delta(u) - \frac{1}{2}\Gamma(0)\big[q_\Delta(t)^2+q_\Delta(t')^2\big] + \Gamma(t-t')q_\Delta(t)q_\Delta(t') + \frac{1}{2}\int_{t'}^{t}\mathrm{d}u\;\Gamma(t-u)\big[q_\Delta(t)\dot{q}_\Delta(u)-q_\Delta(u)\dot{q}_\Delta(t)\big] + \frac{1}{2}\int_{t'}^{t}\mathrm{d}u\;\Gamma(t'-u)\big[q_\Delta(u)\dot{q}_\Delta(t')-q_\Delta(t')\dot{q}_\Delta(u)\big]. \tag{A.8}$$

For $t - t' \gg \Omega^{-1}$, the last three terms can be neglected and the first can be approximated with a single integral:

$$\frac{1}{2}\int_{t'}^{t}\mathrm{d}s\;q_\Delta(s)\ddot{q}_\Delta(s) = \frac{1}{2}q_\Delta(t)\dot{q}_\Delta(t) - \frac{1}{2}q_\Delta(t')\dot{q}_\Delta(t') - \frac{1}{2}\int_{t'}^{t}\mathrm{d}s\;\dot{q}_\Delta(s)^2. \tag{A.9}$$

By using the identity

$$\frac{1}{2}\big[q_\Delta(t)^2 + q_\Delta(t')^2\big] = q_\Delta(t')^2 + \frac{1}{2}\int_{t'}^{t} \mathrm{d}s\; q_\Delta(s)\dot{q}_\Delta(s) \tag{A.10}$$

equation (A.8) can be rewritten† as

$$\int_{t'}^{t} \mathrm{d}s \int_{t'}^{s} \mathrm{d}u\; q_\Delta(s)\ddot{\Gamma}(s-u)q_\Delta(u) = 2m\gamma\left\{\frac{1}{2}\big[q_\Delta(t)\dot{q}_\Delta(t) - q_\Delta(t')\dot{q}_\Delta(t')\big]\right.$$
$$\left. - \frac{1}{2}\int_{t'}^{t} \mathrm{d}s\; \dot{q}_\Delta(s)^2 - \frac{\Omega}{\pi}q_\Delta(t')^2 - \frac{\Omega}{\pi}\int_{t'}^{t} \mathrm{d}s\; q_\Delta(s)\dot{q}_\Delta(s)\right\}. \tag{A.11}$$

In [39, 40] the first term of equation (A.11) is neglected by observing that it is much smaller than the third. In this case the reduced density matrix operator $\hat{\varrho}(t)$ would undergo a transient change

$$\varrho(q_1, q_2, 0) \to \exp\left\{-\frac{2m\gamma\Omega}{12\pi k_B T}\big[q_1(t') - q_2(t')\big]^2\right\}\varrho(q_1, q_2, 0) \tag{A.12}$$

followed by a Lindblad evolution described by an equation which reduces to (3.36) in the $\hbar \to 0$ and $T \to \infty$ limits. The validity of these findings is, however, questionable. The surface terms neglected in (A.11) are of the same order as the integral of $\frac{1}{2}\dot{q}_\Delta^2$ which is, conversely, maintained.

The existence of a transient change in the evolution of the reduced density matrix merits further comment. If the system and the measurement apparatus are initially non-interacting, a change of $\hat{\varrho}(t)$ at the switching-on of the interaction is plausible. However, as we explained in section 2, this transient cannot be described in the framework of a model, such as the bath of harmonic oscillators, in which the system and the measurement apparatus are always in interaction. We must limit our considerations to a non-transient evolution and, correspondingly, assume that the system and the meter are correlated from the beginning.

The drawbacks of [39, 40] have also been underlined recently in [41]. As in [21] and the present work, the authors of [41] assume an initial correlation between the system and the environment.

Appendix B. Non-selective measurements on a free-particle cat state

Let us consider a free quantum particle which, at the beginning of the measurement, is in the superposition (cat) state

$$|\psi(t')\rangle = N\big[|p_1q_1\rangle + |p_2q_2\rangle\big] \tag{B.1}$$

where $|p_iq_i\rangle$, $i = 1, 2$, are two coherent states (5.24) with $\omega_0 = 0$ and the normalization factor is

$$N = \frac{1}{\sqrt{2\big[1 + \exp(-C_{p_1-p_2q_1-q_2})\big]}}. \tag{B.2}$$

The initial state (B.1) has no classical counterpart. The $\hbar \to 0$ limit of the corresponding Wigner function

$$W(p, q, t') = N^2\left\{W_{p_1q_1}(p, q) + W_{p_2q_2}(p, q) + W_{\frac{p_1+p_2}{2}\,\frac{q_1+q_2}{2}}(p, q)\right.$$
$$\left.\times 2\cos\left[\frac{p}{\hbar}(q_1 - q_2) - \frac{p_1 - p_2}{\hbar}\left(q - \frac{q_1 + q_2}{2}\right)\right]\right\} \tag{B.3}$$

† Note that our definition of γ is twice that used in [39, 40].

where $W_{p'q'}(p, q)$ is given by equation (5.37), does not exist. The situation changes during the measurement. In the case of a non-selective process, the system is described by a Wigner function $W(p, q, t)$ which can be evaluated exactly by solving equation (5.39) with the initial condition (B.3)†. The result is

$$
\begin{aligned}
W(p, q, t) = N^2 \Bigg\{ & W_{p_1 q_1}(p, q, t) + W_{p_2 q_2}(p, q, t) \\
& + W_{\frac{p_1+p_2}{2}\,\frac{q_1+q_2}{2}}(p, q, t) \exp\big(-C_{p_1-p_2 q_1-q_2} + \Sigma_{p_1-p_2 q_1-q_2}(t)\big) \\
& \times 2\cos\left[\frac{p_1+p_2}{2\hbar}(q_1-q_2) + \Upsilon_{p_1-p_2 q_1-q_2}(t)\left(p - \frac{p_1+p_2}{2}\mathrm{e}^{-\gamma(t-t')}\right)\right. \\
& \left. + \Phi_{p_1-p_2 q_1-q_2}(t)\left(q - \frac{q_1+q_2}{2} - \frac{p_1+p_2}{2m\gamma}\big(1-\mathrm{e}^{-\gamma(t-t')}\big)\right)\right]\Bigg\}
\end{aligned}
\tag{B.4}
$$

where

$$
\begin{aligned}
W_{p'q'}(p, q, t) = & \frac{1}{2\pi\sqrt{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}} \\
& \times \exp\Bigg\{ -\frac{C_{xx}(t)}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}\left[q - q' - \frac{p'}{m\gamma}\big(1-\mathrm{e}^{-\gamma(t-t')}\big)\right]^2 \\
& + \frac{C_{xy}(t)}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}\left[q - q' - \frac{p'}{m\gamma}\big(1-\mathrm{e}^{-\gamma(t-t')}\big)\right]\left[p - p'\mathrm{e}^{-\gamma(t-t')}\right] \\
& - \frac{C_{yy}(t)}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}\left[p - p'\mathrm{e}^{-\gamma(t-t')}\right]^2\Bigg\}
\end{aligned}
\tag{B.5}
$$

$$
\Sigma_{p'q'}(t) = \frac{C_{xx}(t)C^y_{p'q'}(t)^2 - C_{xy}(t)C^x_{p'q'}(t)C^y_{p'q'}(t) + C_{yy}(t)C^x_{p'q'}(t)^2}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}
\tag{B.6}
$$

$$
\Upsilon_{p'q'}(t) = \frac{2C_{yy}(t)C^x_{p'q'}(t) - C_{xy}(t)C^y_{p'q'}(t)}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}
\tag{B.7}
$$

and

$$
\Phi_{p'q'}(t) = \frac{2C_{xx}(t)C^y_{p'q'}(t) - C_{xy}(t)C^x_{p'q'}(t)}{4C_{xx}(t)C_{yy}(t) - C_{xy}(t)^2}
\tag{B.8}
$$

are given in terms of the coefficients

$$
C_{xx}(t) = \hbar m\gamma\left\{\frac{1}{2}\frac{\hbar}{m\gamma\sigma_q^2}\left(\frac{1}{4} + \frac{\sigma_{pq}^4}{\hbar^2}\right)\mathrm{e}^{-2\gamma(t-t')} + \frac{1}{2}\frac{k_\mathrm{B}T}{\hbar\gamma}\left[1 - \mathrm{e}^{-2\gamma(t-t')}\right]\right\}
\tag{B.9}
$$

$$
\begin{aligned}
C_{xy}(t) = \hbar\Bigg\{ & \frac{\hbar}{m\gamma\sigma_q^2}\left(\frac{1}{4} + \frac{\sigma_{pq}^4}{\hbar^2}\right)\left[1 - \mathrm{e}^{-\gamma(t-t')}\right]\mathrm{e}^{-\gamma(t-t')} + \frac{\sigma_{pq}^2}{\hbar}\mathrm{e}^{-\gamma(t-t')} \\
& + \frac{k_\mathrm{B}T}{\hbar\gamma}\left[1 - 2\mathrm{e}^{-\gamma(t-t')} + \mathrm{e}^{-2\gamma(t-t')}\right]\Bigg\}
\end{aligned}
\tag{B.10}
$$

† By Fourier transforming equation (5.39) with respect to p and q, one obtains a quasi-linear partial differential equation which can be solved by standard methods [42].

$$C_{yy}(t) = \frac{\hbar}{m\gamma}\left\{\frac{1}{2}\frac{m\gamma\sigma_q^2}{\hbar}\left[1+\frac{\sigma_{pq}^2}{m\gamma\sigma_q^2}\left[1-\mathrm{e}^{-\gamma(t-t')}\right]\right]^2+\frac{1}{8}\frac{\hbar}{m\gamma\sigma_q^2}\left[1-\mathrm{e}^{-\gamma(t-t')}\right]^2\right.$$
$$+\frac{k_\mathrm{B}T}{\hbar\gamma}\left[\gamma(t-t')-\frac{3}{2}+2\mathrm{e}^{-\gamma(t-t')}-\frac{1}{2}\mathrm{e}^{-2\gamma(t-t')}\right]$$
$$\left.+\frac{1}{16}\frac{\hbar\gamma}{k_\mathrm{B}T}\gamma(t-t')\right\} \tag{B.11}$$

$$C^x_{p'q'}(t) = p'\frac{\sigma_{pq}^2}{\hbar}\mathrm{e}^{-\gamma(t-t')} - q'\frac{\hbar}{\sigma_q^2}\left(\frac{1}{4}+\frac{\sigma_{pq}^4}{\hbar^2}\right)\mathrm{e}^{-\gamma(t-t')} \tag{B.12}$$

$$C^y_{p'q'}(t) = p'\frac{\sigma_q^2}{\hbar}\left\{1+\frac{\hbar}{m\gamma\sigma_q^2}\frac{\sigma_{pq}^2}{\hbar}\left[1-\mathrm{e}^{-\gamma(t-t')}\right]\right\}$$
$$-q'\left\{\frac{\hbar}{m\gamma\sigma_q^2}\left(\frac{1}{4}+\frac{\sigma_{pq}^4}{\hbar^2}\right)\left[1-\mathrm{e}^{-\gamma(t-t')}\right]+\frac{\sigma_{pq}^2}{\hbar}\right\}. \tag{B.13}$$

The indices x and y stand for the Fourier variables conjugated with p and q, respectively. Each term of (B.4) is localized within a phase-space region whose size grows with time. When this size has became much larger than $\sigma_p\sigma_q \sim \hbar$, we can write

$$\int \mathrm{d}p'\,\mathrm{d}q'\; W(p',q',t)\; W_{p'q'}(p,q) \simeq W(p,q,t) \tag{B.14}$$

and identify the weights $\overline{\delta(p-p(t))\delta(q-q(t))}$ of equation (5.36) with $W(p,q,t)$ in agreement with [30].

Let us now turn to the classical limit of (B.4). First, we note that $W_{p'q'}(p,q,t)$ is the time evolution of the Wigner function (5.37) corresponding to the coherent state $|p'q'\rangle$. Its classical limit exists and is given by an expression $W^{\mathrm{cl}}_{p'q'}(p,q,t)$ identical to (B.5) with $C_{xx}(t)$, $C_{xy}(t)$ and $C_{yy}(t)$ replaced by

$$C^{\mathrm{cl}}_{xx}(t) = \lim_{\hbar\to 0} C_{xx}(t) = \tfrac{1}{2}mk_\mathrm{B}T\left[1-\mathrm{e}^{-2\gamma(t-t')}\right] \tag{B.15}$$

$$C^{\mathrm{cl}}_{xy}(t) = \lim_{\hbar\to 0} C_{xy}(t) = \frac{k_\mathrm{B}T}{\gamma}\left[1-2\mathrm{e}^{-\gamma(t-t')}+\mathrm{e}^{-2\gamma(t-t')}\right] \tag{B.16}$$

$$C^{\mathrm{cl}}_{yy}(t) = \lim_{\hbar\to 0} C_{yy}(t) = \frac{k_\mathrm{B}T}{m\gamma^2}\left[\gamma(t-t')-\frac{3}{2}+2\mathrm{e}^{-\gamma(t-t')}-\frac{1}{2}\mathrm{e}^{-2\gamma(t-t')}\right]. \tag{B.17}$$

The function $W^{\mathrm{cl}}_{p'q'}(p,q,t)$ is the phase-space probability density obtained by solving the Fokker–Plank equation (2.14) with initial condition $W^{\mathrm{cl}}_{p'q'}(p,q,t') = \delta(p-p')\delta(q-q')$.

Concerning the interference term in equation (B.4), we have $\Sigma_{p'q'}(t') = C_{p'q'}$, $\Upsilon_{p'q'}(t') = q'/\hbar$ and $\Phi_{p'q'}(t') = -p'/\hbar$ so that, as previously noted, the $\hbar \to 0$ limit of this term does not exist at $t = t'$ due to the undamped oscillation of the cosine. On the other hand, for $t > t'$ since $\Sigma_{p'q'}(t) = \mathcal{O}(\hbar^{-1})$ while $C_{p'q'} = \mathcal{O}(\hbar^{-3/2})$ we have an exponentially damping term which allows to obtain

$$\lim_{\hbar\to 0} W(p,q,t) = \tfrac{1}{2}\left[W^{\mathrm{cl}}_{p_1q_1}(p,q,t)+W^{\mathrm{cl}}_{p_2q_2}(p,q,t)\right]. \tag{B.18}$$

From a physical point of view, this limit is equivalent to a macroscopic one in which $|p_1-p_2|/\sigma_p$ and/or $|q_1-q_2|/\sigma_q$ become infinitely large so that $C_{p_1-p_2q_1-q_2}$ diverges. In particular, this is obtained by taking the mass m of the particle infinitely large.

Finally, we note that due to the condition $\hbar\gamma \ll k_B T$ the Wigner function $W_{p'q'}(p, q, t)$ approaches the classical phase-space probability density $W^{\rm cl}_{p'q'}(p, q, t)$ on a time scale $(\hbar/\gamma k_B T)^{1/2} \ll \gamma^{-1}$†. On the other hand, the functions $\Sigma_{p'q'}(t)$, $\Upsilon_{p'q'}(t)$ and $\Phi_{p'q'}(t)$ vanish for $t \to \infty$. The long-time limit of equation (B.4) is therefore

$$W_\infty(p, q, t) = N^2 \Bigg\{ W^{\rm cl}_{p_1 q_1}(p, q, t) + W^{\rm cl}_{p_2 q_2}(p, q, t) + W^{\rm cl}_{\frac{p_1+p_2}{2}\,\frac{q_1+q_2}{2}}(p, q, t) \times {\rm e}^{-C_{p_1-p_2 q_1-q_2}} 2\cos\left[\frac{p_1+p_2}{2\hbar}(q_1-q_2)\right]\Bigg\} \qquad \text{(B.19)}$$

and never coincides with the classical limit (B.18).

References

[1] Schrödinger E 1935 *Naturwissenschaften* **23** 807 (Engl. transl. Trimmer J P 1980 *Proc. Am. Phil. Soc.* **124** 323)
Schrödinger E 1935 *Naturwissenschaften* **23** 823 (Engl. transl. Trimmer J P 1980 *Proc. Am. Phil. Soc.* **124** 323)
Schrödinger E 1935 *Naturwissenschaften* **23** 844 (Engl. transl. Trimmer J P 1980 *Proc. Am. Phil. Soc.* **124** 323)
[2] Einstein A 1969 *The Born–Einstein Letters* ed M Born (New York: Walker) p 213
[3] Joos E and Zeh H D 1985 *Z. Phys.* B **59** 223
[4] Zurek W H 1981 *Phys. Rev.* D **24** 1516
Zurek W H 1982 *Phys. Rev.* D **26** 1862
[5] Cini M 1983 *Nuovo Cimento* B **73** 27
[6] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[7] Wheeler J A and Zurek W H (ed) 1983 *Quantum Theory and Measurement* (Princeton, NJ: Princeton University Press)
[8] Belavkin V P, Hirota O and Hudson R L (ed) 1955 *Quantum Communications and Measurement* (New York: Plenum)
[9] Presilla C, Onofrio R and Tambini U 1996 *Ann. Phys., NY* **248** 94 and references therein
[10] Bohr N 1931 *Nature* **128** 691
[11] Haake F and Zukowski M 1993 *Phys. Rev.* A **47** 2506
[12] Wan K K and Harrison F 1994 *Found. Phys.* **24** 831
[13] van Fraassen B C 1991 *Quantum Mechanics* (Oxford: Clarendon)
[14] Diósi L 1988 *Phys. Lett.* **129A** 419
[15] Diósi L 1988 *Phys. Lett.* **132A** 233
[16] Gisin N and Percival I C 1992 *J. Phys. A: Math. Gen.* **25** 5677
Gisin N and Percival I C 1992 *Phys. Lett.* **167A** 315
Gisin N and Percival I C 1993 *J. Phys. A: Math. Gen.* **26** 2233
[17] Percival I C 1994 *J. Phys. A: Math. Gen.* **27** 1003
[18] Ghirardi G C, Rimini A and Weber T 1987 *Phys. Rev.* D **34** 470
[19] Halliwell J J and Zoupas A 1995 *Phys. Rev.* D **52** 7294
[20] Caldeira A O and Leggett A J 1983 *Physica* **121A** 587
[21] Patriarca M 1996 *Nuovo Cimento* B **111** 61
[22] Jona Lasinio G and Presilla C 1996 *Phys. Rev. Lett.* **77** 4322
[23] Kubo R, Toda M and Hashitsume N 1985 *Statistical Physics II* (Berlin: Springer)
[24] Feynman R P and Vernon F L 1963 *Ann. Phys., NY* **24** 118
[25] Arnold L 1974 *Stochastic Differential Equations: Theory and Applications* (New York: Wiley)
[26] Lindblad G 1976 *Commun. Math. Phys.* **48** 119

† The time scale in question can be defined by the conditions $C^{\rm cl}_{xx}(t)/C_{xx}(t) = \frac{1}{2}$, $C^{\rm cl}_{xy}(t)/C_{xy}(t) = \frac{1}{2}$ and $C^{\rm cl}_{yy}(t)/C_{yy}(t) = \frac{1}{2}$, the value of these ratios being 0 at $t = t'$ and monotonically approaching 1 for $t \to \infty$. For $\hbar\gamma/k_B T \ll 1$ and $\gamma(t - t') \ll 1$, each one of the above conditions is an equation for the unknown $\gamma(t - t')(\hbar\gamma/k_B T)^{-1/2}$ whose solution is of the order of unity.

Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821
[27] Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
[28] Yaffe L G 1982 *Rev. Mod. Phys.* **54** 407
[29] Dekker H and Valsakumar M C 1984 *Phys. Lett.* **104A** 67
[30] Halliwell J and Zoupas A 1996 The post-decoherence density matrix propagator for quantum Brownian motion *E-print archives* quant-ph/9608046
[31] Feynman R P and Hibbs H R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[32] Spiller T P and Ralph J F 1994 *Phys. Lett.* **194A** 235
[33] Steimle T, Alber G and Percival I C 1995 *J. Phys. A: Math. Gen.* **28** L491
[34] Brun T A, Percival I C and Schack R 1996 *J. Phys. A: Math. Gen.* **29** 2077
Brun T A, Gisin N, O'Mahony P F and Rigo M 1996 *E-print archives* quant-ph/9608038
Gisin N, Brun T A and Rigo M 1996 *E-print archives* quant-ph/9611002
[35] Brune M *et al* 1996 *Phys. Rev. Lett.* **77** 4887
[36] Strunz W T and Percival I C 1997 The semiclassical limit of quantum state diffusion—a phase space approach *Preprint* QMW-PH-97-3
[37] Hu B L, Paz J P and Zhang Y 1992 *Phys. Rev.* D **45** 2843
[38] Ambegaokar V 1991 *Ber. Bunsenges. Phys. Chem.* **95** 400
[39] Diósi L 1993 *Europhys. Lett.* **22** 1
[40] Diósi L 1993 *Physica* **199A** 517
[41] Tameshtit A and Sipe J E 1996 *Phys. Rev. Lett.* **77** 2600
[42] See, for instance, Zwillinger D 1992 *Handbook of Differential Equations* (New York: Academic)